Announcements, Expectations, and Stock Returns with Asymmetric Information*

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Abstract: The fact that consensus forecast revisions positively predict forecast errors is often interpreted as evidence against rational expectations. We further document that stock market reactions to forecast revisions negatively predict stock market reactions to forecast errors. In a dynamic noisy rational expectations model with periodic macroeconomic announcements, we show that the average of individuals' rational beliefs is not Bayesian given all publicly available information -- it underweights new information relative to Bayes' rule. In addition, stock price is more sensitive to noise when public information is imprecise and less so when macroeconomic news is informative. Our calibration quantitatively accounts for the seeming 'under-reaction' of consensus beliefs and 'over-reaction' of stock prices.

Keywords: Expectations Formation, Macroeconomic Announcement, Rational Inattention, Noisy and Asymmetric Information.

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1 Introduction

Stock market investors are in a constant battle to separate noise from signal. Many factors influence prices. Most of these factors are unobserved by individual investors, and most will turn out to have only transitory effects. Given its importance to understanding asset prices, models of signal extraction have long been a workhorse in financial economics, ranging from the classic static models of Grossman and Stiglitz (1980) and Hellwig (1980), to the more complex dynamic models of Wang (1993; 1994), to more recent endogenous information models reviewed in Veldkamp (2011). However, this previous literature has neglected one key feature of real world information environments, namely, their non-stationarity. Information enters the market periodically, via recurrent macroeconomic data releases. This paper shows that time variation in the underlying information structure creates testable predictions, which have yet to be examined.

Following recent work by Coibion and Gorodnichenko (2015) (henceforth CG15), we exploit an interesting feature of the Survey of Professional Forecasters (SPF). The SPF asks participants to report a *sequence* of quarterly GDP growth forecasts for the next four quarters. These forecasts are submitted roughly in the middle of each quarter. The difference between the current forecast for this quarter and the previous forecast submitted last quarter provides a measure of forecast revisions concerning the current quarter's growth rate. Roughly six weeks later, the actual (initial) estimate is announced by the Bureau of Economic Analysis (BEA).[1] As in CG15, we focus on consensus forecasts, constructed as the cross-sectional mean forecast. The key advantage of using these survey measures of forecast revisions is that they allow us to evaluate the market's response to beliefs without committing to a particular statistical model that supposedly generates these beliefs.

CG15 were interested in the potential role of information frictions in explaining apparent violations of the Rational Expectations Hypothesis. They show that consensus forecast revisions positively predict subsequent forecast errors, not only for GDP but for several other macroeconomic variables. They go on to show that models featuring either 'sticky' or 'noisy' information can explain this predictability. This paper extends the work of CG15 by examining the stock market's responses to these revisions and forecast errors. It shows that market reactions to revisions *negatively* predict subsequent reactions to forecast errors. Although seemingly contradictory, the same underlying mechanism as in CG15 is at work. Between announcements, noise has the opportunity to accumulate, since the underlying state has yet to be revealed. Sometimes this noise accumulation is favorable, sometimes it's unfavorable. If it's favorable, it will likely have produced a positive forecast revision for this quarter, and accordingly, the market will have reacted positively. Later, once the true state is revealed, the market must correct itself, and prices decline. Everything works in reverse if the noise accumulation has been unfavorable. Either way, market responses to forecast revisions negatively predict subsequent responses to forecast errors.[2]

[1] As discussed below, the actual sequence of forecasts and announcements is a little more complicated. In practice, the BEA provides monthly initial and revised estimates of quarterly GDP growth.

[2] The mechanism here is similar to the 'wisdom after the fact' models of Romer (1993) and Caplin and Leahy (1994). The key difference is that in those models fixed costs or externalities prevent fundamental information from getting revealed to the market until a critical threshold is reached. In contrast, here it is noise that accumulates, and

After documenting this empirical regularity, we show that an extension of Wang's (1993) dynamic noisy Rational Expectations model (NREE) can account for it *quantitatively.* we extend Wang's model in two ways. First, the information structure in Wang's model is exogenous. Informed agents know the underlying state, whereas uninformed agents must learn about it. Here we assume heterogeneous channel capacities constrain *both* agents information processing efforts. Hence, even relatively informed agents remain unsure about the underlying state until it is announced. This enables our model to capture the sort of information rigidities reported in CG15. Second, the information structure in Wang (1993) is also time invariant. Here it is time-varying. Following announcements, both agents obtain full information. Imperfect and asymmetric information emerges endogenously in between announcements due to the heterogeneous channel capacities. We show that a time-varying information structure can explain a seemingly puzzling empirical result, namely, that realized volatility does not change following announcements. Although increased information following announcements would by itself increase volatility, the endogenous reduction in price sensitivity offsets this.

As in Wang (1993), we show that the equilibrium price is a linear function of the underlying state. However, here the pricing coefficients are time-varying, and must solve a system of Ordinary Differential Equations subject to boundary conditions at the announcement dates. This quasi-analytical solution makes it easy to generate model-implied time paths. Using these time paths, we replicate the regressions we conducted using the actual data. We find that reasonable parameter values can match the empirically observed regression coefficients.

Related Literature. This paper is closely related to the recent literature on endogenous information in macroeconomics and finance. Veldkamp (2011) provides a survey. Sims (2003) first introduced Rational Inattention based on information processing constraints. Mackowiak and Wiederholt (2012) formulate information processing as a costly choice depending on limited liability.[3] A related literature attributes information rigidities to sticky information, such as Mankiw and Reis (2002). Agents update their beliefs infrequently, but acquire full information once they do so. As a result, sticky information by itself cannot explain why price responses to forecast revisions predict price responses to forecast errors. For this, it is essential that the underlying state is unknown by *all* agents until the announcement occurs.

This paper also contributes to the literature on the role of asymmetric information in asset markets. Easley, Kiefer, and O'Hara (1997) and Easley, Hvidkjaer, and O'hara (2002) analyze information-based trading and emphasize the importance of asymmetric information in affecting stock returns. Goldstein and Yang (2017) provide a review of information disclosure in financial markets via public announcements. It is worth noting the the heterogeneity emphasized in this paper is endogenously generated by information processing constraints. This is different from the large literature on heterogeneous priors, such as Basak (2000; 2005), Scheinkman and Xiong (2003). Here

the threshold is exogenously determined by data announcement dates.

[3]Huang and Liu (2007) investigate how investors choose their attention frequency to periodic news; Zhang (2006) empirically documents the role of information uncertainty in influencing stock returns.

incomplete information comes from one type of investor having superior channel capacity, which generates endogenous heterogeneous forecasts. Crucially, this information asymmetry is eliminated periodically following macroeconomic announcements, thus producing the predictability of price responses to the information revelation.

The remainder of the paper is organized as follows. Section 2 provides empirical evidence on forecast revisions and forecast errors. It first confirms the results of CG15, that GDP forecast revisions positively predict GDP forecast errors. It then shows that stock price responses to forecast revisions *negatively* predict price responses to forecast errors. Section 3 illustrates the basic intuition behind this result using a simple 2-period model. Section 4 then develops a complete general equilibrium continuous-time model featuring periodic macroeconomic announcements. Section 5 provides a quantitative analysis of the model. It shows that data generated from the model can explain the empirical evidence documented in Section 2. Finally, Section 6 concludes by discussing a few possible extensions. A technical Appendix contains robustness checks, proofs, and derivations.

2 Empirical Evidence

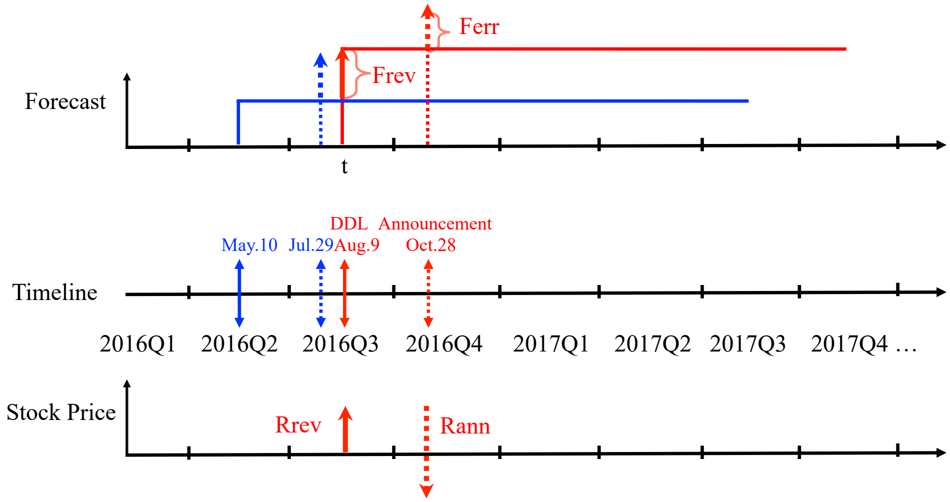
In this section, we first follow CG15 by showing that revisions of SPF forecasts of quarterly GDP growth positively predict subsequent forecast errors. As in CG15, we focus on consensus forecasts, defined as the cross-sectional mean forecast. we then go on to examine the stock market's reactions to both forecast revisions and forecast errors, and show the opposite pattern arises: market reactions to forecast revisions partly reflect noise, and so negatively predict market responses to forecast errors. To clarify the timeline and measurements illustrated in this section, see Figure 2.1 for a simple example.

2.1 Data and Measurements

Forecasts. Following CG15 and most of the literature on expectations, we use historical survey data from the Survey of Professional Forecasters (SPF). We focus on forecasts of GDP growth (SPF abbreviation: RGDP). SPF is a quarterly survey containing approximately 40 professional forecasters, beginning in 1968Q4. Since 1990 it has been run by the Federal Reserve Bank of Philadelphia. Panelists come largely from the business world and Wall Street, spanning different sectors (e.g. banks, consulting firms, universities, private firms, etc). Each forecaster is asked to forecast at horizons from the current quarter (t) to four quarters later ($t+4$). The data are reported at both the individual-level and the consensus-level, computed as the cross-sectional mean from the individual-level forecasts at a point in time.[4] After 1990Q2, the survey has been conducted in the second month of the quarter, and the deadline for submitting forecasts is around the middle of the

[4] The cross-sectional mean could change due to a change in forecaster composition. CG15 only include forecasters that participate in two consecutive surveys and find robust results.

Figure 2.1: A Simple Example



This figure gives a simple illustration of the underlying timing. To be specific, let's focus on the third quarter of 2016 (in red lines). After the BEA's announcement for 2016Q2 real GDP growth rate on 2016.7.29, the SPF survey was distributed to the SPF forecasters. The submission deadline was 2016.8.9. Panelists must forecast five quarters, from the current quarter 2016Q3 to 2017Q4 (Note, the forecasts over the five periods are not necessarily the same). This also took place last quarter (2016Q2, blue lines), where panelists needed to forecast from 2016Q2 to 2017Q3, and submitted before 2016.5.10. Suppose on May.10, the cross-sectional average forecast for 2016Q3 GDP growth was 2.35%. One quarter later, on Aug.9, panelists forecasted 2016Q3 again, say 2.8%. Compared to last quarter's forecast, they revised up $Frev = 0.45\%$. The stock market return on Aug.9 was $Rrev = 5.96$ basis points. On Oct.28, the BEA announced an advance (first) estimate of 2016Q3 realized GDP growth rate of 2.9%. Hence, the forecast error $Ferr = 0.1\%$. And the overnight return turned out to be $Rann$ = -1.41 basis points.

survey month.[5,6]

Define the consensus forecast revision ($Frev_t$) and forecast error ($Ferr_t$) as:

$$Frev_t = \mathbb{F}_t x_t - \mathbb{F}_{t-1} x_t \tag{1}$$

$$Ferr_t = x_t - \mathbb{F}_t x_t \tag{2}$$

where $\mathbb{F}_t$ denotes the average time t forecasts across panelists. Forecast revisions are calculated as the difference between the current quarter's consensus forecast ($\mathbb{F}_t$) of this quarter's GDP growth rate x_t, and last quarter's forecast ($\mathbb{F}_{t-1}$) of the current quarter's GDP growth rate (forecasting one quarter ahead within the information set at $t-1$). Hence, forecast revisions reflect the new information obtained and processed by agents from $t-1$ to t. Forecast errors are defined as the difference between the realization of x_t (initial release about quarter t GDP growth rate, announced at the beginning of $t+1$), and the forecasts made at t.[7,8]

[5]The detailed deadline date and news release date can be found in: https://www.philadelphiafed.org/-/media/research-and-data/real-time-center/survey-of-professional-forecasters/spf-release-dates.txt?la=en.

[6]The survey is distributed after the release of advance GDP estimates, and first gets published (open to the public) around one week later than the submission deadline. Since this paper mainly focuses on stock market reactions to stock market participants' forecast revisions, the data publication day does not matter because it could not represents the time when those forecasters, as the representative of stock market participants, revise their beliefs.

[7]Note that the forecast is done before the true data is realized.

[8]CG15 use forecasts of year-on-year annual growth rate from quarter t to $t+4$. Since this paper is interested in

Announcements. We collect GDP announcement dates from the Bureau of Economic Analysis (BEA)'s website, where they report the annualized quarterly growth rate at the end of each month.[9] GDP growth rate announcements are made monthly, so that each quarter contains three announcements: advance (first), second, and third estimate. For example, in April the advance estimate for Q1 GDP growth rate is released, followed by a second estimate of the same Q1 GDP growth rate in May, and a third estimate given in June. We focus on the advance estimates for two reasons. First, the advance estimates are believed to include the most information, and resolve most of the uncertainty. Second, the subsequent revisions may not reflect the initial investors' reactions to the surprise in GDP growth rate announcements. Therefore, in this paper, the forecast revision at quarter-t is associated with the realization of quarter-t GDP growth rate announced at $t+1$, and forecast errors are calculated using advance estimates.

Stock Market Returns. To measure the stock market's reaction to expectation revisions, we use close-to-close returns on survey submission deadline days $Rrev_t = \frac{close_t - close_{t-1day}}{close_{t-1day}}$, and overnight returns on GDP announcement days $Rann_t = \frac{open_t - close_{t-1day}}{close_{t-1day}}$. We use realized close-to-close returns on deadline dates for three reasons. First, although information arrives continuously, and beliefs are accordingly being continuously revised, the submission days provide a direct, model-free, snapshot of this continuous revision process. As a robustness check, we also calculate returns using surrounding dates.[10,11] Second, we use overnight returns at advance estimate announcement days since GDP growth rate is released at 8:30 a.m., before the stock market opens. Overnight returns capture the stock market's reaction to forecast errors more accurately because they include all the price response to the GDP announcement. Third, instead of using the S&P 500, we calculate returns based on the SPDR S&P 500 ETF (SPY) dataset, which is available from January 1993. We do this because the S&P 500 calculates its opening price at 9:31 a.m., when many stocks are not open. As a result, the opening price for the S&P 500 is often the same as the previous trading day's closing price, which produces many zero overnight returns when using S&P 500 data. In contrast, the SPY is calculated based on S&P 500 futures, which is always open by 9:31 a.m.[12] Hence, we use $Rrev_t$ to measure the stock market's reaction to forecast revisions and $Rann_t$ to measure the stock market's reaction to forecast errors upon announcements.

stock market reactions to expectations formation process, we only look at adjacent forecast revision and examine the stock market reaction to both forecast revision and error about the same quarter t. In the appendix, we show that the result is robust when examining CG15 year-on-year annual forecast revision and forecast error.

[9]https://www.bea.gov/data/gdp/gross-domestic-product.

[10]We exclude those observation dates lying on non-trading days.

[11]One may argue that the revision has been started right after the last submission day and forecast revisions defined in equation (1) should reflect the cumulative revisions in response to the new obtained information between two adjacent forecast quarters ($t-1$ to t). It is also possible that panelists start revisions after they receive the survey (there are no exact dates for when the surveys were distributed, but approximately within one to two weeks before the deadline dates). In the appendix, we show that the results remain robust if considering close-to-close return between current and last quarter submission deadline days, or close-to-close returns between the submission deadline day and one week before it.

[12]For the robustness check, we have also shown that the results hold when using the return data from NASDAQ. The results are even stronger by using daily returns from Kenneth French's website instead of close-to-close returns from SPY.

Table 2.1: Summary Statistics

Variable	Mean (%)	S.D.(%)	N	Time
$Rrev_t$	0.012	1.35	84	1993Q1-2018Q4
$Rann_t$	0.15	0.67	104	1993Q1-2018Q4
$Frev_t$	-0.27	0.09	200	1968Q4-2018Q4
$Ferr_t$	0.08	0.13	200	1968Q4-2018Q4
x_t	2.41	0.21	200	1968Q4-2018Q4

This table reports summary statistics for the main variables used in the empirical tests. $Rrev_t$ is the close-to-close returns on SPF submission deadline days. $Rann_t$ is the overnight return on BEA advance estimate announcement days. Forecast errors ($Ferr_t$) equal to realized x_t minus forecasts at t. Forecast revision ($Frev_t$) is defined as the difference between forecast of the GDP growth rate at quarter t and forecast of the same rate made at quarter $t-1$. Realized value x_t uses BEA advance estimate of quarter-t GDP growth rate announced at quarter-$t+1$. All returns exclude observations on non-trading days.

Table 2.1 summarizes the data used in the empirical analysis. Several things are worth noting. First, there are no abnormal returns observed on non-announcement days. It is reasonable to believe that on average there is no special news announced or events happened on revision days. Second, the average announcement return is 15 basis points, while the standard deviation is only 0.67%. This is consistent with a large literature on the macroeconomic announcement premium. Investors face uncertainty and must be compensated for it.[13] Third, one can see that professional forecasters do not have significant forecasting biases, at least at 1-quarter ahead horizons. Therefore, we assume they are marginal investors who can generally represent the stock market participants.[14]

2.2 Empirical Tests

Main Results. First, applying the methodology developed by CG15, specify the following regression,

$$Ferr_t = \alpha + \beta_F Frev_t + \varepsilon_t. \tag{3}$$

Second, the key relationship in this paper, between stock price reactions to forecast revisions and price reactions to forecast errors, can be characterized by:

$$Rann_t = \alpha + \beta_P Rrev_t + \varepsilon_t, \tag{4}$$

where β_P is the OLS estimate obtained from regressing the announcement day returns on previous revision day returns in the same quarter. In addition, we control for forecast revision $Frev_t$, forecast

[13] Ai and Bansal (2018) develop a framework for understanding the macroeconomic announcement premium based on generalized risk sensitivity preferences.

[14] Stark et al. (2010) analyzes the accuracy of forecasts, and finds SPF forecasts outperform benchmark projections from univariate autoregressive time-series models at short horizons. The special survey of analyzing the panelists' forecasting methods shows that, "20 of 25 respondents said they use a combination of mathematical/computer models *plus* subjective adjustments to that model in reporting their projections."

error $Ferr_t$, and changes in GDP growth rate defined as the difference between current and last quarter advance estimate of GDP growth rates: $\Delta x_t = x_t - x_{t-1}$.[15]

The sign of βs indicates patterns of 'under-' and 'over-' reaction relative to the representative agent FI-RE benchmark, as noted by CG15.[16] If βs are insignificantly different from zero, the forecasters form expectations based on FI-RE. However, if βs are significantly positive, those forecasters do not response sufficiently to new information, and vice versa. For example, $\beta_F > 0$ implies that if the forecasters revise up their beliefs in response to news, ex post, they make positive errors because they did not revise up enough. CG15 interpret the positive β_F as measuring the degree of information rigidity. Table 2.2 displays the main results.

Table 2.2: Tests of GDP Growth Rate Expectations Formation Process in Stock Market

	(1)	(2)	(3)	(4)	(5)	(6)
	$Rann_t$	$Ferr_t$	$Rrev_t$		$Rann_t$	
$Rrev_t$	-0.206**					-0.199**
	(0.084)					(0.086)
$Frev_t$		0.393**	0.336*		-0.128**	-0.051
		(0.170)	(0.205)		(0.055)	(0.048)
$Ferr_t$				0.032		0.035
				(0.054)		(0.060)
Δx_t						0.009
						(0.032)
Constant	0.145***	0.179	0.098	0.148***	0.128**	0.135**
	(0.057)	(0.131)	(0.001)	(0.062)	(0.059)	(0.055)
N	84	199	84	103	104	83
R^2	0.169	0.070	0.066	0.004	0.028	0.184

This table reports coefficient estimates of regressing first row variables on first column variables using Ordinary Least Squares following CG15. Newey-West (lagged 5) standard errors are in parentheses. Note: $^*p < 0.1;^{**} p < 0.05;^{***} p < 0.01$.

Column (1) demonstrates the paper's main finding: the stock market appears to over-react to new information, in the sense that stock returns on revision days negatively predict announcement day returns. In order to understand why this is puzzling, let's look at the results step-by-step. First, column (2) shows that consensus forecast revisions positively predict forecast errors. This confirms the results of CG15 that consensus forecasts apparently under-react to news.[17] Second, as you would expect, upward revisions of GDP growth increase stock returns (see column (3)). Third,

[15]The results remain robust if further controlling for one and two lagged level of realized GDP growth rate.

[16]Note that the terms of under- and over-reaction we use throughout the paper are interpreted as departures from the representative agent FI-RE benchmark. While some literature interpreted those terms as rejections for rational expectations assumption, this paper is totally based on rational expectations framework.

[17]CG15 attribute this result as the evidence of information rigidities. They show that the empirical evidence coincide with two types of models, NREE models with Rational Inattention (Sims, 2003; Mackowiak and Wiederholt, 2009) and sticky information models (Mankiw and Reis, 2002). In this paper, the model specification corresponds to the first type.

column (4) shows that, although not significant, a positive surprise to forecasted GDP growth increases the overnight return on announcement days. The above three facts imply that investors revise up their expectations about future GDP growth rate when they receive positive news on the revision day. This moves up the stock return on the same day. When the true value is realized upon announcement, it turns out that forecasters made more mistakes in previous estimation and did not revise up enough. In another words, the ex post realized GDP growth rate is even higher than previously expected. Since a positive surprise should imply an increase in stock return, it is natural to think that the announcement day return should also increase to reflect the positive surprise. However, column (1) illustrates the opposite: the stock return at announcement day *drops* significantly.[18] Column (5) displays the negative predictability of forecast revision itself to stock market reaction to forecast errors. However, this fact disappears and is replaced by the predictability of stock market reaction to forecast revisions (column (6)).[19]

These two empirical results are based on direct measures of economic agents' expectations, and do not depend on any auxiliary assumptions about economic models. Both point to a rejection of the representative-agent FI-RE hypothesis, which could be driven by any of the following three assumptions: representative-agent, full-information or rational expectations. CG15 attribute the under-reaction to a rejection of full-information and the supportive evidence consistent with the presence of information rigidities. Interestingly, evidence from the stock market, first documented in this paper, points to the importance of heterogeneous agents with noisy and asymmetric information, but still under the framework of rational expectations. In particular, in order to reconcile these two empirical facts, there must be at least one agent who has superior information over the other. However, this information advantage is still noisy, coming from public signal extraction instead of from private and inaccessible information.

3 A Two-Period Model

In this section, we present a two-period NREE model to illustrate this paper's basic intuition and key economic mechanism.

3.1 The FIRE Representative Agent Benchmark

Consider an economy with a measure **1** of agents. Agents only live for two periods and maximize their expected utility $u(C_0) + \mathbb{E}[u(C_1)]$,[20] with identical CARA preferences: $u(C) = -e^{-C}$. Each agent i is endowed with q_i units of a Lucas tree that pays (x_0, x) on dates 0 and 1, where the payoff on date 1 is uncertain with $x \sim N\left(\bar{x}, \rho_x^{-1}\right)$. The total endowment in the economy is given, with $\theta = \int q_i di$. There are two types of securities, a risk-free bond with interest rate normalized to zero, and a risky asset with the market price P.

[18] This effect is even more significant ($\beta = -0.3, t = -3.23$) when using close-to-close return at announcement days.
[19] All the results remain robust if excluding the recent financial crisis period.
[20] For simplicity, the time discount rate and risk aversion are assumed to be 1.

The optimization problem for each agent can be characterized as:

$$\max_{C_{i,0},C_{i,1},\alpha_i,B_i} u\left(C_{i,0}\right)+\mathbb{E}\left[u\left(C_{i,1}\right)\right] \tag{5}$$
$$s.t.\ C_{i,0}+B_i+\alpha_i P=q_i\left(x_0+P\right)$$
$$C_{i,1}=\alpha_i x+B_i,$$

where $C_{i,0}, C_{i,1}$ denote the consumption at dates 0 and 1. B_i and α_i denote portfolio shares agent i holds in risk free bonds and risky assets, respectively. The initial wealth of each agent is the product of his/her initial endowment q_i and the initial value of the risky asset, x_0+P.

A competitive equilibrium is defined as follows: (i) Given the price P, agent i's optimal decisions $\{C_{i,0}, C_{i,1}, \alpha_i, B_i\}$ solve the objective function (5). (ii) The goods market clears: $\int C_{i,0}di = x_0\theta$; $\int C_{i,1}di = x\theta$. (iii) Bonds are in zero net supply: $\int B_i di = 0$; and (iv) The stock market clears: $\int \alpha_i di = \theta$.

The optimal solution yields agent i's demand function for risky assets:

$$\alpha_i=\frac{\mathbb{E}\left(x\right)-P}{\mathrm{Var}\left(x\right)}, \tag{6}$$

which gives $\alpha_i=\bar{x}-\rho_x P$. This implies the familiar result that an agent's demand does not depend on the initial endowment, such that $\alpha_i=\theta$ for all i. Therefore, the equilibrium price is given by

$$P=\bar{x}-\frac{\theta}{\rho_x}.$$

In this simple framework with perfect information, the representative agent owns the entire Lucas tree. Price is decreasing in total endowment supply with the sensitivity of $1/\rho_x$.

3.2 The Static NREE Model

Assume at date 0, agent i observes a noisy signal of x. In this case, information is heterogeneous. The precision of this signal ρ_ε reflects the agent's channel capacity:

$$s_i=x+\varepsilon_i,\ \ \varepsilon_i\sim N(0,\rho_\varepsilon^{-1}),$$

where ε_i is independent of x and normally distributed with mean 0 and variance ρ_ε^{-1}.

Agents have rational expectations and update their beliefs according to Bayes' rule based on their own signals. Agent i's posterior mean and variance are:

$$\begin{aligned}\mathbb{E}\left[x\right|x+\varepsilon_i] &= \frac{1}{\rho_x+\rho_\varepsilon}\left[\rho_x\bar{x}+\rho_\varepsilon\left(x+\varepsilon_i\right)\right],\\ \mathrm{Var}\left[x\right|x+\varepsilon_i] &= \frac{1}{\rho_x+\rho_\varepsilon}.\end{aligned}$$

Plugging in Equation (6) immediately gives: $\alpha_i=\rho_x\bar{x}+\rho_\varepsilon\left(x+\varepsilon_i\right)-\left(\rho_x+\rho_\varepsilon\right)P$. Finally, using the market clearing condition, $\int \alpha_i di = \theta$, the price can be derived as

$$P = \frac{\rho_x}{\rho_x + \rho_\varepsilon}\bar{x} + \frac{\rho_\varepsilon}{\rho_x + \rho_\varepsilon}x - \frac{1}{\rho_x + \rho_\varepsilon}\theta. \tag{7}$$

Since the total asset endowment θ is known to the agent, observing price will fully reveal the information about the fundamental cash flow x.

To prevent the price from being fully revealing in equilibrium, assume θ is a random variable and normally distributed with mean 0 and variance $1/\rho_\theta$, $\theta \sim N(0, \rho_\theta^{-1})$. As usual, θ can represent noise or liquidity trades. We simply refer to it as noise. Notice that θ is mean-reverting to 0, meaning that noise must converge to zero in the long run. As a result, agents do not care about it, even though price would respond to it in the short-run.

Since agents only care about fundamental cash flows x and the equilibrium price contains information about it, agents learn from the observed price. Note that observing the equilibrium price is equivalent to observing the following public signal of x:

$$s_p = x + e,$$

where $e \sim N\left(0, \rho_e^{-1}\right)$ is independent of x. Agent i rationally learns about x based on his/her own signal s_i and the public signal s_p extracted from the price, with the posterior mean and variance

$$\begin{aligned} \mathbb{E}\left[x \middle| x + \varepsilon_i, x + e\right] &= \frac{1}{\rho_x + \rho_\varepsilon + \rho_e}\left[\rho_x \bar{x} + \rho_\varepsilon \left(x + \varepsilon_i\right) + \rho_e \left(x + e\right)\right], \\ \mathrm{Var}\left[x \middle| x + \varepsilon_i, x + e\right] &= \frac{1}{\rho_x + \rho_\varepsilon + \rho_e}. \end{aligned}$$

The risky asset demand for agent i is: $\alpha_i = \rho_x \bar{x} + \rho_\varepsilon \left(x + \varepsilon_i\right) + \rho_e \left(x + e\right) - \left(\rho_x + \rho_\varepsilon + \rho_e\right) P$. Using the market clearing condition, the price can be derived as:

$$P = \frac{\rho_x}{\rho_x + \rho_\varepsilon + \rho_e}\bar{x} + \frac{\rho_\varepsilon + \rho_e}{\rho_x + \rho_\varepsilon + \rho_e}x - \frac{\rho_\varepsilon + \rho_e}{\rho_x + \rho_\varepsilon + \rho_e}\frac{1}{\rho_\varepsilon}\theta, \tag{8}$$

where $\rho_e = \rho_\varepsilon^2 \rho_\theta$. See Appendix 6.2 for the proof.

Define the sensitivity of price with respect to noise as $\phi_\theta = \frac{\rho_\varepsilon + \rho_e}{\rho_x + \rho_\varepsilon + \rho_e}\frac{1}{\rho_\varepsilon} = \frac{1 + \rho_\varepsilon \rho_\theta}{\rho_x + \rho_\varepsilon + \rho_\varepsilon^2 \rho_\theta} \geq 0$. Similarly, the sensitivity of price with respect to the fundamental is defined as $\phi_x = \frac{\rho_\varepsilon + \rho_e}{\rho_x + \rho_\varepsilon + \rho_e} \leq 1$. At announcements, there is no heterogeneous information. This reduces to a representative agent framework. Observing the signal is equivalent to observing the true value of x. When $\rho_\varepsilon \to \infty$, x is known, price fully reveals θ. At the same time, $\phi_x \to 1$ and $\phi_\theta \to 0$. Prices react more to fundamentals and, more importantly, less to noise.

This simple two-period model provides the following important intuition: prices are more sensitive to noise when they are not fully revealing. However, given any precise information about fundamentals, price becomes fully revealing. As a result, equilibrium prices are less sensitive to non-fundamental noise shocks.

3.3 Under-Reaction

This section uses the above two-period model to rationally explain why consensus forecast revisions appear to under-react to news. In another word, the predictability of consensus forecast errors from forecast revisions. First, assume the economy has one representative agent who tries to learn about unobserved fundamental x. Suppose the agent has the prior distribution of $N\left(\mu, \sigma^2\right)$. Moreover, the agent can observe a noisy signal $s = x + \varepsilon$, where $\varepsilon \sim N\left(0, \sigma_\varepsilon^2\right)$.

Suppose the agent is a 'rational' learner, so that Bayes' rule gives the posterior mean as:

$$\mathbb{E}\left[x \middle| s\right] = \bar{\lambda}\mu + \left(1 - \bar{\lambda}\right) s,$$

where $\bar{\lambda} = \frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma_\varepsilon^2}}$.

Suppose there exists another operator $\mathcal{E}$ under some arbitrary rules, which could be different from the expectation operator $\mathbb{E}$ under Bayes rule. The forecast revision and forecast error can be calculated as:

$$\begin{aligned}
\mathcal{E}\left[x \middle| s\right] - \mu &= \left(1 - \lambda\right)\left(x - \mu\right) + \left(1 - \lambda\right)\varepsilon, \\
x - \mathcal{E}\left[x \middle| s\right] &= \lambda\left(x - \mu\right) - \left(1 - \lambda\right)\varepsilon.
\end{aligned}$$

This implies,

$$\text{Cov}\left\{x - \mathcal{E}\left[x \middle| s\right], \mathcal{E}\left[x \middle| s\right] - \mu\right\} = \lambda\left(1 - \lambda\right)\sigma^2 - \left(1 - \lambda\right)^2 \sigma_\varepsilon^2.$$

Hence, $\text{Cov}\left\{x - \mathcal{E}\left[x \middle| s\right], \mathcal{E}\left[x \middle| s\right] - \mu\right\} \geq 0$ as long as $\lambda \geq \bar{\lambda}$, implying that forecast revisions negatively predict forecast errors. The intuition is that when agents are Bayesians, forecast revisions should not be able to predict forecast errors. If they did, agents would revise their beliefs to incorporate this predictability in order to improve their forecasts. However, when there exists some other non-Bayesian rules which overweight priors and underweight new information, forecast revision will appear to under-react to news.

To see how it works under heterogeneous information models with rational learning, let's extend the representative agent framework a bit further. Assume there are two independent signals $s_1 = x + \varepsilon_1$, and $s_2 = x + \varepsilon_2$ in the economy, where ε_1 and ε_2 are i.i.d with $N\left(0, \sigma_\varepsilon^2\right)$.

First, suppose information is complete, and that one agent can observe both signals, Bayes' rule gives:

$$\mathbb{E}\left[x \middle| s\right] = \bar{\lambda}\mu + \frac{1}{2}\left(1 - \bar{\lambda}\right) s_1 + \frac{1}{2}\left(1 - \bar{\lambda}\right) s_2,$$

where $\bar{\lambda} = \frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma_\varepsilon^2} + \frac{1}{\sigma_\varepsilon^2}}$.

Second, assume there are two agents in the economy with heterogenous information, in the sense that agent 1 only observes s_1, whereas agent 2 only observes s_2. Each agent i does rational learning without observing the other's signal. The posterior of agent i is derived as:

$$\mathbb{E}_i\left[x|\, s_i\right] = \lambda\mu + \lambda s_i,$$

where $\lambda = \frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2}+\frac{1}{\sigma_\varepsilon^2}}$. The consensus forecast is simply the average of the two agents' forecasts:

$$\frac{1}{2}\left\{\mathbb{E}_1\left[x|\, s_1\right] + \mathbb{E}_2\left[x|\, s_2\right]\right\} \;\; = \;\; \lambda\mu + \frac{1}{2}\left(1-\lambda\right)s_1 + \frac{1}{2}\left(1-\lambda\right)s_2.$$

Clearly, $\lambda > \bar{\lambda}$. Therefore, the consensus forecast revisions positively predict forecast errors. The intuition is that with heterogeneous channel capacities, agents observe heterogeneous information, which produces endogenously heterogeneous forecasts. Although each agent rationally updates the belief using his/her own information set, neither agent can observe the other's signal. Aggregating over both agents underweights the unobserved signals due to the incomplete information. This captures the sort of information rigidities reported in CG15. It is important to notice that, the consensus belief could not captured by any representative agent's belief. Heterogeneity is crucial under this rational framework.

4 The Full Model

This section develops a continuous time NREE model with predetermined periodic macroeconomic announcements. There are two reasons for building a dynamic continuous time model. First, stock prices respond to announcements immediately, but aggregate consumption or dividends do not. Continuous time methods can capture this instantaneous change upon announcements. Second, a dynamic model allows us to quantitatively simulate the economy, and thereby quantitatively account for the empirical facts presented in Section 2.

More specifically, consider a general equilibrium model under noisy and asymmetric information. Define the fundamentals as everything related to cash flows, while noise or non-fundamentals do not.[21] A shock is called fundamental if it has a permanent effect on output or cash flows. In contrast, noise shocks are only transitory, and are uncorrelated with fundamentals. In this model, fundamentals are the dividend level and latent growth rate, and noise is given by the stochastic risky equity supply. There are two types of agents, informed and uninformed investors. Assume both investors observe the dividend level and stock price, but not the latent dividend growth rate. The informed investor obtains an additional noisy signal, reflecting his superior information processing capacity (Sims, 2003). The predetermined announcements are assumed to reveal true realizations of dividend growth rates. As a result, asymmetric information disappears, and the economy reduces (temporarily) to a representative single agent economy.

[21]Essentially, those fundamentals could be stochastic processes capturing exogenous changes in technology, preferences, endowments or government policy. Non-fundamentals could be the total equity supply, unobserved discount rates from others, etc.

4.1 The Economy

4.1.1 The Physical Environment

Let D_t, x_t, θ_t be the exogenous dividend level, latent dividend growth, and total amount of noise (risky equity supply). The physical environment can be written as follows:

$$dD_t = (x_t - \kappa D_t)\, dt + \sigma_D dB_{D,t}, \tag{9}$$

$$dx_t = b\,(\bar{x} - x_t)\, dt + \sigma_x dB_{x,t}, \tag{10}$$

$$d\theta_t = -a\theta_t dt + \sigma_\theta dB_{\theta,t}, \tag{11}$$

where $dB_{D,t}$, $dB_{x,t}$, and $dB_{\theta,t}$ are independent standard Brownian motions. The parameters σ_D, σ_x, and σ_θ measure conditional volatility over an incremental unit of time with respect to D_t, x_t, and θ_t. When $\kappa = 0$, the dividend process becomes non-stationary (containing a unit-root). When $\kappa > 0$, this process is stationary and mean-reverting, known as an Ornstein-Uhlenbeck (OU) process. With CARA preferences, it is important that dividends and prices be stationary. To ensure stationarity of dividends and prices, we assume $\kappa = 1$. In this case, κ measures the degree of mean reversion. In contrast to standard noisy information models, where some "informed" investors have private information and know the true value of fundamentals, the state variable x_t is assumed to be unobserved to all investors. a and b are positive constants governing the persistence of dividends and noise. The noise/non-fundamental process specified in (11) has a zero long-run mean, so θ_t represents the deviation of current non-fundamentals from the long-run stationary level. Even though in the short-run noise fluctuates and induces temporary price changes, in the long run its effect must converge to zero.

4.1.2 Information Structure and Pricing

Assume there is a fraction ω of uninformed investors and a fraction $(1 - \omega)$ of informed investors. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space. $\mathcal{F}^u$, $\mathcal{F}^i$ denote the uninformed and informed investor's information sets, respectively. Assume all investors know the structure of the model and underlying parameter values. They rationally update their beliefs based on their own information sets, but due to Rational Inattention, never fully observe the true states.

Noisy information due to Rational Inattention. Following Kasa (2006) and Luo (2016), we assume the informed investor observes an additional noisy signal s_t about x_t due to rational inattention motive:

$$ds_t = x_t dt + \sigma_s dB_{s,t}. \tag{12}$$

where σ_s is the signal volatility and $dB_{s,t}$ is an independent Brownian motion.

This specification captures the idea that there always exists some freely available public information about x_t. Due to rational inattention, e.g. they may have a time constraint or limited information processing capacity, investors may not choose to process such information. The representative uninformed investor captures this type of agent. On the other hand, investors can also choose to obtain a noisy signal by allocating attention to this public information, e.g. devoting more effort to collect and process information. These provide the micro-foundations for informed investors. The amount of attention allocation or information processing capacity determines the precision of the signal $1/\sigma_s$ in equation (12).[22] Therefore, informed investors have all the information of the uninformed, but they also have an additional noisy signals about fundamentals.

Information structure on non-announcement days. On non-announcement days, both investor types update their beliefs about fundamentals x_t using realized dividends D_t, and more importantly, the stock price P_t. Neither can observe the stochastic non-fundamental noise θ_t directly. While the equilibrium price fully reveals θ_t to the informed investor, it reveals a combination of θ_t and the informed investor's signal to uninformed investor. Therefore, by rational learning from stock prices, uninformed investor cannot distinguish between noise and the informed investor's beliefs about fundamentals. Hence, the information structure is summarized as follows: $\mathcal{F}^u = \{D_\tau, P_\tau; \tau \leq t\}$, $\mathcal{F}^i = \{D_\tau, P_\tau, s_\tau, \theta_\tau; \tau \leq t\}$. Obviously, $\mathcal{F}^u \subset \mathcal{F}^i \subset \mathcal{F}$.

Define the posterior mean and variance of informed investors as $\mathbb{E}\left[x_t \mid \mathcal{F}^i\right] = \hat{x}_t$, $\mathbb{E}\left[\left(\hat{x}_t - x_t\right)^2 \mid \mathcal{F}^i\right] = \hat{q}(t)$. Further, define the posterior mean of uninformed as $\mathbb{E}\left[x_t \mid \mathcal{F}^u\right] = \check{x}_t$, $\mathbb{E}\left[\hat{x}_t \mid \mathcal{F}^u\right] = \tilde{x}_t$, $\mathbb{E}\left[\theta_t \mid \mathcal{F}^u\right] = \tilde{\theta}_t$, and the posterior variance-covariances as $\tilde{q}_{ij}(t) = \mathbb{E}\left[\left(\vartheta_i(t) - \tilde{\vartheta}_i(t)\right)\left(\vartheta_j(t) - \tilde{\vartheta}_j(t)\right) \mid \mathcal{F}^u\right]$, where $\vartheta_t = \left[\begin{array}{ccc} x_t & \hat{x}_t & \theta_t \end{array}\right]^\top$ includes their unobservables. In addition, by tower property, $\mathbb{E}\left[\hat{x}_t \mid \mathcal{F}^u\right] = \mathbb{E}\left[\mathbb{E}\left[x_t \mid \mathcal{F}^i\right] \mid \mathcal{F}^u\right] = \check{x}_t = \tilde{x}_t$ *a.s.* This implies that the posterior mean estimation of x_t from uninformed investor must be the same no matter he/she estimates directly from the true state x_t, or learn from the posterior of the informed $\hat{x}_t$ indirectly.

Information structure on announcement days. Assume there are pre-determined announcements at times nT $(n = 1, 2, \ldots)$. Within each announcement cycle, $t \in [0, T]$, denote $\cdot^-$ and $\cdot^+$ as a variable right before and after the announcement.

Announcements provide information, and play two roles in affecting the market equilibrium. First, announcements fully reveal the true fundamentals x_t to all investors, as a result, $\hat{x}_T^+ = \tilde{x}_T^+ = x_T$ and $t = 0$, representing a restart of another announcement cycle. Second, upon announcements, price is fully revealing and asymmetric information is removed between the two types of agents. This further reveals θ_t to the uninformed, gives $\tilde{\theta}_T^+ = \theta_T$.

Equilibrium market price. Generally, the equilibrium stock price P_t depends on all the state variables $D_{t,}, \theta_t, \hat{x}_t$, and $\tilde{x}_t$. Following the standard approach of solving NREE models, solving for the equilibrium pricing function proceeds as follows: first guess a functional form using undetermined

[22] Han, Kasa, and Luo (2019) introduce heterogeneous information processing capacities between institutional investors and households to motivate households' delegation to financial intermediaries. Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) document cyclical allocation of attention of fund managers based on Rational Inattention. Luo (2016) shows the observational equivalence between a fixed information processing capacity and an exogenous specified signal-to-noise ratio.

coefficients, then use it to solve the agents' optimization problems, and then compute the coefficients by imposing market clearing conditions.

To begin with, conjecture the equilibrium price takes the following linear form:

$$P_t = \phi(t) + \phi_D D_t + \phi_\theta(t)\theta_t + \phi_x(t)\hat{x}_t + \phi_\Delta(t)\tilde{x}_t. \tag{13}$$

where $\phi_\theta(t) < 0$, $\phi_x(t) \geq 0$ and $\phi_\Delta(t) \geq 0$ are time-varying sensitivities of price with respect to noise θ_t, and informed and uninformed investors' posterior beliefs about fundamentals $\hat{x}_t$ and $\tilde{x}_t$, respectively. The crucial element in this paper is allowing these sensitivities to be time-varying. Unlike the steady state models of Grossman and Stiglitz (1980) and Wang (1993), time-varying sensitivities are necessary to generate the predictability of price reactions to expectations formation. The reason is, stock price reacts more strongly to noise on non-announcement days due to asymmetric information, but becomes fully revealing to everyone upon announcements. Therefore, $\phi_\theta(t)$ must change over time to reflect different price sensitivities. More precisely, suppose on a non-announcement day, there has been a negative shock in θ_t which drives down the stock price. Since price is less sensitive to θ_t upon announcements ($\phi_\theta(t)$ is smaller), shocks to θ_t are not perceived as riskier as that on non-announcement days. Price must go up to reflect this reduction in uncertainties. Therefore, stock prices on non-announcement and announcement days are negatively correlated given the same shocks.

Given the conjectured price function, it is clear that the equilibrium price reveals θ_t to informed investors ($\{D_t, \hat{x}_t, \tilde{x}_t\} \in \mathcal{F}^i$ implies $\mathcal{F}^i = \{D_\tau, s_\tau, \hat{x}_\tau, \tilde{x}_\tau, \theta_\tau; \tau \leq t\}$). On the other hand, since $\{D_t, \tilde{x}_t\} \in \mathcal{F}^i$, observing price would not fully reveal true values of θ_t and $\hat{x}_t$ to uninformed investors. However, they effectively observe a combination of them: $\phi_x(t)\hat{x}_t + \phi_\theta(t)\theta_t$. Define $\zeta_t \equiv \phi_x(t)\hat{x}_t + \phi_\theta(t)\theta_t$, uninformed investors' information set could equivalently be written as $\mathcal{F}^u = \{D_\tau, \tilde{x}_\tau, \zeta_\tau; \tau \leq t\}$. This further implies $\zeta_t = \mathbb{E}[\zeta_t \mid \mathcal{F}^u] \Leftrightarrow \phi_x(t)\hat{x}_t + \phi_\theta(t)\theta_t = \phi_x(t)\tilde{x}_t + \phi_\theta(t)\tilde{\theta}_t$. Therefore, one can view $\tilde{\theta}_t$ as depending on existing state variables, which can be excluded from the price function:

$$\tilde{\theta}_t = \theta_t + \frac{\phi_x(t)}{\phi_\theta(t)}(\hat{x}_t - \tilde{x}_t) = \theta_t + \frac{\phi_x(t)}{\phi_\theta(t)}\Delta_t, \tag{14}$$

where $\Delta_t \equiv \hat{x}_t - \tilde{x}_t$ is the difference in estimated fundamentals between informed and uninformed agents. The following proposition summarizes the equilibrium pricing function:

Proposition 1. *In the interior, $t \in [(n-1)T^+, nT^-]$, $n = 1, 2, \ldots$, the equilibrium price function takes the following linear form, measured with respect to informed and uninformed investor's information sets, $\mathcal{F}^i, \mathcal{F}^u$, respectively:*

$$P_t = \phi(t) + \phi_D D_t + \phi_\theta(t)\theta_t - \phi_\Delta(t)\Delta_t + \phi_y\hat{x}_t, \tag{15}$$

$$= \phi(t) + \phi_D D_t + \phi_\theta(t)\tilde{\theta}_t + \phi_y\tilde{x}_t, \tag{16}$$

where $\phi_D = \frac{1}{\kappa+r}$, $\phi_y = \phi_x(t) + \phi_\Delta(t) = \frac{\phi_D}{b+r}$, and $\phi_\theta(t) < 0$.

Proof. See Appendix 6.5 for the proof. □

For the informed investor, a positive shock to θ_t reduces the equilibrium stock price and increases its expected return ($\phi_\theta(t) < 0$ and $e_\theta(t) > 0$). Intuitively, increases in θ_t make the the agent's portfolio riskier. Therefore the expected return must increase because investors require compensation for risk. Similarly, $\tilde{\theta}_t$ characterizes the uninformed investors' beliefs about θ_t. Whenever their *perceived* equity risk increases, their expected return also increases. Moreover, the informed investor has an informational advantage, Δ_t, coming from the additional noisy signal. When $\Delta_t < 0$, the uninformed investor is unduly optimistic. This drives up the stock price since $\phi_\Delta(t) > 0$. However, the informed investor knows that the future announcement will correct for this "over-estimation" by uninformed investors. Hence, the future price must drop to reflect this correction. Therefore, the informed investor's expected return is negatively correlated with the estimation difference ($e_\Delta(t) > 0$).

4.1.3 Filtering Problem

The informed investor tries to learn x_t using his/her information set $\mathcal{F}^i$ while the uninformed investor tries to learn $\vartheta_t = \begin{bmatrix} x_t & \hat{x}_t & \theta_t \end{bmatrix}^\top$ based on $\mathcal{F}^u$. The following lemma summarizes the posterior conditional distributions by applying Liptser and Shiryaev (2001) Theorem 10.3:

Lemma 1. *In the interior, $t \in [(n-1)T^+, nT^-]$, $n = 1, 2, \ldots$, the Kalman filter equations for the informed investor's conditional mean and variance are given by:*

$$d\hat{x}_t = b(\bar{x} - \hat{x}_t)\,dt + \frac{\hat{q}(t)}{\sigma_D} d\hat{B}_{D,t} + \frac{\hat{q}(t)}{\sigma_s} d\hat{B}_{s,t} \tag{17}$$

$$d\hat{q}(t) = \left[\sigma_x^2 - 2b\hat{q}(t) - \hat{\sigma}\hat{q}^2(t)\right] dt \tag{18}$$

where $\hat{\sigma} = \frac{1}{\sigma_D^2} + \frac{1}{\sigma_s^2}$. $d\hat{B}_{D,t} = \frac{1}{\sigma_D}\left[dD_t - (\hat{x}_t - \kappa D_t)\,dt\right]$, and $d\hat{B}_{s,t} = \frac{1}{\sigma_s}(ds_t - \hat{x}_t dt)$ are innovations corresponding to (9) and (12), respectively.

For uninformed investors, the filtering equations for conditional mean and variance can be written as

$$d\tilde{x}_t = b(\bar{x} - \tilde{x}_t)\,dt + h_{11}(t)\,d\tilde{B}_{D,t} + h_{12}(t)\,d\tilde{B}_{\zeta,t}, \tag{19}$$

$$d\tilde{\theta}_t = -a\tilde{\theta}_t dt + h_{21}(t)\,d\tilde{B}_{D,t} + h_{22}(t)\,d\tilde{B}_{\zeta,t}, \tag{20}$$

where the coefficients $h_{11}(t)$, $h_{12}(t)$, $h_{21}(t)$, $h_{22}(t)$ and innovations $d\tilde{B}_{D,t}$, $d\tilde{B}_{\zeta,t}$ are given in Appendix 6.3. The conditional variance-covariance matrix $\tilde{q}(t)$ satisfies the equivalent conditions and stochastic processes defined in Equations (47) and (49) in Appendix 6.2.

Proof. See Appendix 6.3 for the proof. □

Note, Equation (47) implies that the uninformed investor's posterior variance of x_t is a summation of the informed investor's posterior variance of x_t, and his/her posterior variance of $\hat{x}_t$. As mentioned earlier, even though the posterior mean has to be the same whether learning from the true state x_t directly or from informed investor's posterior $\hat{x}_t$ indirectly, the posterior variances are different. Intuitively, uninformed investor knows that learning would reduce the estimation error. As a result, compared to learning directly from the true state, learning indirectly from informed investor's posterior would reduce the amount of estimation variance already deducted by informed investor.

Furthermore, the distribution of the estimation difference Δ_t between informed and uninformed investors can be obtained immediately:

$$d\Delta_t = -a_\Delta(t)\,\Delta_t dt + \sigma_{\Delta D}(t)\,d\hat{B}_{D,t} + \sigma_{\Delta s}(t)\,d\hat{B}_{s,t} + \sigma_{\Delta\theta} dB_{\theta,t}, \tag{21}$$

where $a_\Delta(t)$, $\sigma_{\Delta D}(t)$, $\sigma_{\Delta s}(t)$, and $\sigma_{\Delta\theta}$ are given Appendix 6.4 Note that Δ_t is an OU process, mean-reverting to zero under the informed investor's information set. This shows that, in the long run, the estimation error is only temporary without announcements. The reason is, both investors are rational and are estimating the same underlying process for x_t. Even though uninformed investors have less estimation precision, in the long run, the estimation difference between two types of investors converges to zero with more information. More interestingly, macroeconomic announcements reveal the precise information periodically, reducing Δ_t to zero whenever $t = nT$.

Define the excess return Q_t under informed and uninformed investors' information sets $\mathcal{F}^i$, $\mathcal{F}^u$ as Q_t^i and Q_t^u respectively. Under the proposed functional form for the equilibrium price in Proposition 1, their stochastic processes could be formulated as follows:

$$\begin{aligned} dQ_t^i &= \left[e_0(t) + e_\theta(t)\,\theta_t + e_\Delta(t)\,\Delta_t\right] dt + b_D^i(t)\,d\hat{B}_{D,t} + b_s^i(t)\,d\hat{B}_{s,t} + b_\theta^i(t)\,dB_{\theta,t}, && (22)\\ dQ_t^u &= \left[e_0(t) + e_\theta(t)\,\tilde{\theta}_t\right] dt + b_D^u(t)\,d\tilde{B}_{D,t} + b_\zeta^u(t)\,d\tilde{B}_{\zeta,t}, && (23) \end{aligned}$$

where the coefficients are given in Appendix 6.5. Hence, the expected returns are $\mathbb{E}\left[dQ_t^i \mid \mathcal{F}^i\right]/dt = e_0(t) + e_\theta(t)\,\theta_t + e_\Delta(t)\,\Delta_t$, and $\mathbb{E}\left[dQ_t^u \mid \mathcal{F}^u\right]/dt = e_0(t) + e_\theta(t)\,\tilde{\theta}_t$, respectively.

It is clear that both investors' expected excess returns depend on noise. For the informed investor, a positive shock to θ_t reduces the equilibrium stock price and increases the expected return ($\phi_\theta(t) < 0$ and $e_\theta(t) > 0$). Intuitively, increases in θ_t make the equity riskier, which then increases the expected return because investors require compensation for risk. Similarly, $\tilde{\theta}_t$ characterizes the uninformed investor's beliefs about θ_t. Whenever the total equity they *perceived* become riskier, their expected return also increases. Moreover, the informed investor has information privilege over Δ_t, coming from the additional noisy signals obtained due to Rational Inattention. When $\Delta_t < 0$, the informed investor's posterior belief about fundamentals is lower than that of the uninformed. This drives up the stock price since $\phi_\Delta(t) > 0$. However, the informed investors know that announcements will eventually correct for this "over-estimation" by uninformed investors. And future price must drop to reflect this correction. Therefore, the informed investor's expected return is negatively correlated with the estimation difference ($e_\Delta(t) > 0$).

4.2 Equilibrium Conditions on Non-Announcement Days

Assume there are pre-determined announcements every period at times nT $(n = 1, 2, \ldots)$. Within each announcement cycle, $t \in [0, T]$. On non-announcement days, investors solve optimization problems in the interior $(0^+ \leq t \leq nT^-, n = 1, 2, \ldots)$. The optimal solutions determine a system of ODEs. At the announcements, investors solve the optimization problems at the boundary ($t = 0 = nT^+$)[23]. Combining the ODEs with the boundary conditions complete the dynamic equilibrium system.

Assume both investors have exponential utilities. This makes asset demands and equilibrium prices independent of the distribution of wealth. A benefit of this specification is that value functions turn out to be quadratic functions of the state variables. This delivers a linear functional form for the equilibrium price, which then verifies the conjectured pricing function in Equation (13).

4.2.1 Optimization Problem on Non-Announcement Days

Now consider an informed investor who has wealth W_t^i, consumption C_t^i, and who invests α_t shares into the risky security. On non-announcement days, the optimization problem for the informed investor in the interior is written as:

$$J\left(t, W^i, \theta, \Delta\right) = \max_{\alpha_t, C_t^i} \mathbb{E}\left[\int_0^{T-t} -e^{-\rho s - C_{t+s}^i} ds + J^-\left(T, W_T^i, \theta_T, \Delta_T\right) \mid \mathcal{F}^i\right],$$

subject to (22), (11), (21), and

$$dW_t^i = \left(W_t^i r - C_t^i\right) dt + \alpha_t dQ_t^i, \tag{24}$$

where ρ is the discount rate and r is the exogenous risk free rate, which we assume is constant.

Similarly, the uninformed investor's optimization problem can be written as:

$$V\left(t, W^u, \tilde{\theta}\right) = \max_{\beta_t, C_t^u} \mathbb{E}\left[\int_0^{T-t} -e^{-\rho s - C_{t+s}^u} ds + V^-\left(T, W_T^u, \tilde{\theta}_T\right) \mid \mathcal{F}^u\right],$$

subject to (23), (20), and

$$dW_t^u = \left(W_t^u r - C_t^u\right) dt + \beta_t dQ_t^u, \tag{25}$$

where W_t^u, C_t^u and β_t are the uninformed investor's wealth, consumption and risky asset portfolio allocation.[24]

The following lemma summarizes the solutions:

[23] Notice that the moment right after each announcement cycle ($t = nT^+$) is the same as the restart of another announcement cycle ($t = 0$).

[24] It is important to note that there may exist multiple equilibria. Investors' value functions could depend on $\hat{x}_t$ and $\tilde{x}_t$. In order to simplify the optimization problem and keep the tractable quadratic formula of the value function, we impose Assumption 1 in Online Appendix 6.9. It can be proved that the unique equilibrium *conditional* on the conjectured value function would finally converge to satisfy the pre-imposed assumption.

Lemma 2. *In the interior ($0^+ \leq t \leq nT^-, n = 1, 2, \ldots$), the informed investor's value function takes the following form:*

$$J\left(t, W^i, \theta, \Delta\right) = -e^{-rW^i - g(t,\theta,\Delta)}, \tag{26}$$

where $g(t, \theta, \Delta)$ is a quadratic form:

$$g(t,\theta,\Delta) = g(t) + g_\theta(t)\,\theta_t + \frac{1}{2}g_{\theta\theta}(t)\,\theta_t^2 + g_\Delta(t)\,\Delta_t + \frac{1}{2}g_{\Delta\Delta}(t)\,\Delta_t^2 + g_{\theta\Delta}(t)\,\theta_t\Delta_t. \tag{27}$$

The uninformed investor's value function takes the following form:

$$V\left(t, W^u, \tilde{\theta}\right) = -e^{-rW^u - f\left(t,\tilde{\theta}\right)}, \tag{28}$$

where $f\left(t, \tilde{\theta}\right)$ is a quadratic form of:

$$f\left(t,\tilde{\theta}\right) = f(t) + f_\theta(t)\,\tilde{\theta}_t + \frac{1}{2}f_{\theta\theta}(t)\,\tilde{\theta}_t^2, \tag{29}$$

where the coefficients are time-varying and satisfy the ODE system defined in Equation(56) and (57) in Appendix 6.6.

The optimal risky asset demand is:

$$\alpha_t = \alpha_0(t) + \alpha_\theta(t)\,\theta_t + \alpha_\Delta(t)\,\Delta_t, \tag{30}$$

$$\beta_t = \beta_0(t) + \beta_\theta(t)\,\tilde{\theta}_t, \tag{31}$$

where $\alpha_0(t)$, $\alpha_\theta(t)$, $\alpha_\Delta(t)$, $\beta_0(t)$ and $\beta_\theta(t)$ are defined in Appendix 6.6.

Proof. See Appendix 6.6 for the derivations. □

4.2.2 Market Equilibrium

Now, combine the policy functions derived in the previous section with market clearing conditions and derive the full coefficients system in the interior. Total risky asset demand must equal to supply in equilibrium, thus, market clearing requires:

$$(1-\omega)\left[\alpha_0(t) + \alpha_\theta(t)\,\theta_t + \alpha_\Delta(t)\,\Delta_t\right] + \omega\left[\beta_0(t) + \beta_\theta(t)\,\tilde{\theta}_t\right] = \theta_t. \tag{32}$$

Combining the above equation with (14) and match the coefficients yields

$$0 = (1-\omega)\,\alpha_0(t) + \omega\beta_0(t), \tag{33}$$

$$1 = (1-\omega)\,\alpha_\theta(t) + \omega\beta_\theta(t), \tag{34}$$

$$0 = (1-\omega)\,\alpha_\Delta(t) + \omega\beta_\theta(t)\,\frac{\phi_x(t)}{\phi_\theta(t)}. \tag{35}$$

Using $\alpha_0(t)$, $\alpha_\theta(t)$ $\alpha_\Delta(t)$, $\beta_0(t)$, $\beta_\theta(t)$ to simplify the coefficients system. This completes the proof of the value function coefficients ODEs system in Lemmas 2 (See Appendix 6.6 for the derivations).

4.3 Equilibrium Conditions at the Announcements

The informed investor's optimization problem at the boundaries (at announcements) is:

$$\begin{aligned} -e^{-rW^{i-}-g(T,\theta_T,\Delta_T)} &= \max_{\alpha_T}\left\{-\mathbb{E}\left[e^{-rW^{i+}-g(0,\theta_T,0)}\mid\mathcal{F}^i\right]\right\} \\ \text{s.t. } W_T^{i+} &= W_T^{i-}+\alpha_T\left(P_T^+-P_T^-\right). \end{aligned}$$

Notice that upon announcements, only price will jump from P_T^- to P_T^+, while the consumption level would not change because of the continuity. Therefore, the law of motion for informed investor's wealth so that the value function would change accordingly, which determines the optimal asset allocation at the boundary. The following corollary summarizes the solution:

Similarly, the uninformed investor's optimization problem at the boundaries (at announcements) is:

$$\begin{aligned} -e^{-rW^{u-}-f\left(T,\tilde{\theta}_T\right)} &= e^{-rW^{u-}}\max_{\beta}\mathbb{E}\left[-e^{-r\beta\left(P_T^+-P_T^-\right)-f(0,\theta_T)}\mid\mathcal{F}^u\right] \\ \text{s.t. } W_T^{u+} &= W_T^{u-}+\beta_T\left(P_T^+-P_T^-\right). \end{aligned}$$

Market clearing requires the total equity demand equals total supply at the announcements: $(1-\omega)\,\alpha+\omega\beta=\theta_T$. This implies

$$(1-\omega)\,\frac{c_0+c_\theta\theta_T+c_\Delta\Delta_T}{r\alpha_q}+\omega\frac{d_0+d_\theta\theta_T+d_\theta\frac{\phi_{x,T}}{\phi_{\theta,T}}\Delta_T}{r\beta_q}=\theta_T. \tag{36}$$

Lemma 3. *At the pre-determined announcement T (equivalent to periodical announcements nT, $n=1,2,\ldots$), the optimal portfolio choice for informed investor is:*

$$\alpha_T=\frac{c_0+c_\theta\theta_T+c_\Delta\Delta_T}{r\alpha_q} \tag{37}$$

where $c_0=\phi(0)-\phi(T)$, $c_\theta=\phi_\theta(0)-\phi_\theta(T)$, $c_\Delta=\phi_\Delta(T)$, and $\alpha_q=\phi_y^2\hat{q}_T$.

The optimal portfolio choice for uninformed investor is:

$$\beta_T=\frac{d_0+d_\theta\theta_T+d_\theta\frac{\phi_{x,T}}{\phi_{\theta,T}}\Delta_T}{r\beta_q} \tag{38}$$

where $d_0=\phi(0)-\phi(T)+f_\theta(0)\,\tilde{q}_{22,T}\phi_{x,T}\left(\phi_{\theta,T}\phi_y-\phi_{x,T}\phi_{\theta,0}\right)$, $d_\theta=\phi_{\theta,T}\left(\phi_{\theta,T}\phi_{\theta,0}+\frac{1-\phi_{\theta,T}^2}{\phi_{x,T}}\phi_y-1\right)$, and $\beta_q=\phi_y^2\hat{q}_T+\left(\phi_{x,T}\phi_{\theta,0}-\phi_y\phi_{\theta,T}\right)^2\tilde{q}_{22,T}$.

The boundary conditions are summarized in Lemma 4, 5 and 6 in Appendix 6.7.

Proof. See Appendix 6.7 for the proof. □

It is easy to see that without time-varying pricing coefficients, the value function would not change upon announcements.

4.4 Stock Market Reactions to Expectations Formation

Upon macroeconomic announcements, the stock market reaction to forecast error can be defined as:

$$P_T^+ - P_T^- = [\phi(0) - \phi(T)] + [\phi_\theta(0) - \phi_\theta(T)]\theta_T + \phi_y(x_T - \hat{x}_T) + \phi_\Delta(T)\Delta_T.$$

Rewritten the price reaction upon announcement into the public information, the following proposition could be derived immediately:

Proposition 2. *Stock market responses to macroeconomic forecasts revisions can be constructed as*

$$P_{t+\delta} - P_t = [\phi(t+\delta) - \phi(t)] + \phi_D(D_{t+\delta} - D_t) + \left[\phi_\theta(t+\delta)\tilde{\theta}_{t+\delta} - \phi_\theta(t)\tilde{\theta}_t\right] + \phi_y(\tilde{x}_{t+\delta} - \tilde{x}_t) \quad (39)$$

where δ denotes the time interval when investors process information and revise their beliefs.

Subsequent market responses to forecast errors are defined as

$$P_T^+ - P_T^- = [\phi(0) - \phi(T)] + \phi_\theta(0)\left(\theta_T - \tilde{\theta}_T\right) + \phi_y(x_T - \tilde{x}_T) + [\phi_\theta(0) - \phi_\theta(T)]\tilde{\theta}_T. \quad (40)$$

Since the investors have made optimal decisions given their own information sets, the revisions should not predict the surprises upon announcements, i.e. $\left(\theta_T - \tilde{\theta}_T\right)$ and $(x_T - \tilde{x}_T)$. Thus, the only predictable component comes from $[\phi_\theta(0) - \phi_\theta(T)]\tilde{\theta}_T$. $\phi_\theta(t)$ measures the sensitivity of stock price to noise. As in the toy model, it is clear that $\phi_\theta(t)$ is a negative and decreasing function and $|\phi_\theta(0)| < |\phi_\theta(T)|$. Upon announcements, price becomes less sensitive to noise. Given $\tilde{\theta}_T$ is public information, this sensitivity change is expected by the market and becomes predictable before the announcement.

The above proposition provides the intuition for why the stock market response to information is time-varying. When public information is imprecise, prices respond more to noise. The existence of asymmetric information amplifies the impact of noise. Since price is sensitive to the dividend growth rate, it provides information about fundamentals. Investors thus extract information from the market price. More noise trading from uninformed investors makes prices less informative about future cash flows. To illustrate the intuition, suppose there is a negative asset supply shock, which increases the stock price. Under information asymmetry, uninformed investors cannot distinguish between stronger fundamentals and a temporary reduction in asset supply by just observing a rise in the stock price. Through rational learning, they attribute the price increase as a weighted average of both fundamental and noise.[25] Therefore, uninformed investors further revise up their beliefs

[25] Chahrour and Jurado (2018) prove that a news representation is observationally equivalent to a noise representa-

about fundamentals, thus price must further rise to clear the market. Since the stock price is more sensitive to noise, it over-reacts to them and deviations from fundamentals accumulate between announcement days. However, upon macroeconomic announcements, price becomes fully revealing, removing the information asymmetry.[26] The market corrects for both the deviations of expectations from the true value of fundamentals, and more importantly, the accumulated market over-reaction to noise. Since price is less sensitive to noise without asymmetric information, market price upon announcements must correct the mistaken over-reactions to noise. Therefore, at announcements, the price reaction to announcements can be negatively predicted by previous price reactions to belief revisions.

5 Quantitative Results

This section combines the interior and boundary conditions obtained in the previous section to solve the ODE system of time-varying price sensitivities and value function coefficients. It then presents a quantitative analysis and demonstrates that the model can account for the stylized empirical facts documented in Section 2.

5.1 Estimates

Table 5.1 contains calibrated and estimated benchmark parameter values. First, preference parameters are chosen to be consistent with the literature: the discount rate $\rho = 0.01$ and the mean relative risk aversion is $\bar{x} = 3.5$. Second, several parameters are calibrated to match outside data. The risk free interest rate $r = 1.5\%$ is calibrated to match US data for the period 1968-2018. $\sigma_d = 1$ is calibrated to match the volatility of the price/dividend ratio. The model produces a mean log price/dividend ratio of 43.6%, compared to 41.8% in the data.[27] $\sigma_x = 0.57$ is calibrated to match return volatility. The model-predicted return volatility is 17.6%, which reasonably matches the empirical annual volatility of 17.3%. Lastly, the remaining parameters $(a, b, \sigma_\theta, \sigma_s)$ are jointly calibrated to match the key results in this paper, i.e., the two regression coefficients reported in Section 2. In general, the model-predicted regression coefficients match the data well. The model implied return response coefficient β_P is -0.18 and the information rigidity coefficient of CG15, β_F, is 0.40. In the data they are -0.21 and 0.39, respectively. The results are discussed in detail in Section 5.3.

5.2 Model Solutions

Under the benchmark parameter values in the Table 5.1, $\phi_D = \frac{1}{\kappa+r} = 0.9852$, $\phi_y = \frac{\phi_D}{b+r} = 4.5824$. Other coefficients are time-varying, and are depicted in Figure 5.1 and Appendix 6.1. The first panel of Figure 5.1 shows that the magnitude of $\phi_\theta(t)$ is smallest at announcements ($t = 0$) and

tion of fundamentals and beliefs.

[26]Announcements provide information and reduce the uncertainties about the underlying fundamentals. The reduction in uncertainty is reflected and priced in the option market, and option-implied variance drops as we could see in the data.

[27]The price/dividend ratio data is from Robert Shiller's webpage.

Table 5.1: Parameters

Para.	Value	Description	Para.	Value	Description
r	0.015	risk free rate	σ_d	1	div. volatility
ρ	0.01	time discount rate	σ_s	0.7	inverse of signal precision
$\bar{x}$	3.5	mean div. growth	σ_x	0.57	unobservable volatility
b	0.2	persistence of div. growth	σ_θ	3	noise volatility
a	0.01	persistence of noise	ω	0.5	fraction of uninformed investor

This table displays annualized parameter values used in the simulation.

$\phi_\theta(nT)$, $n = 1, 2, ...T$. Its magnitude then increases as information endogenously becomes imperfect and asymmetric due to the heterogeneous channel capacities. Uncertainty increases and the stock price becomes more sensitive to noise due to asymmetric information. On announcement days (at time 0), the fundamental is realized (posterior mean of x_t drops to x_t). As a result, uncertainty drops. The stock price is less sensitive to the same amount of noise shocks. Therefore, a previous increase in price predicts a subsequent price drop upon announcements. This gives rise to the negative predictability of market responses to forecast errors.

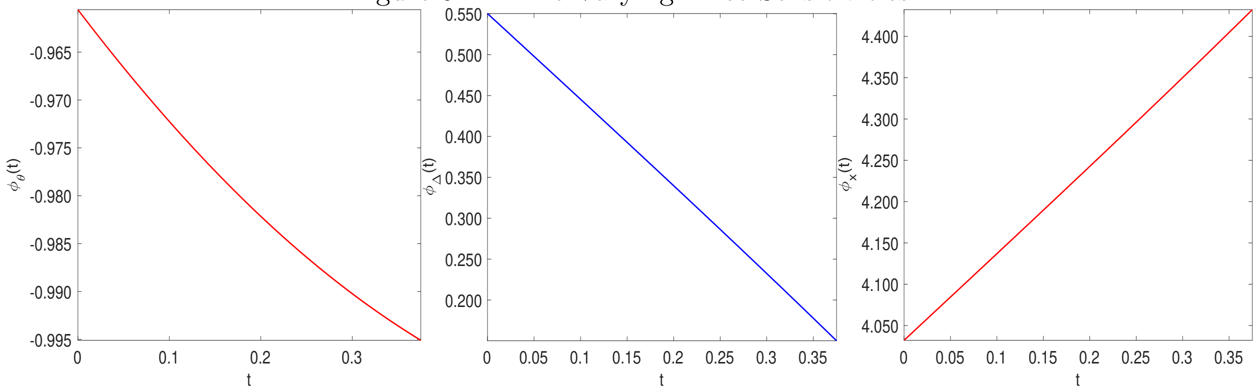
The second and third panel display the time varying sensitivities of $\phi_\Delta(t)$ and $\phi_x(t)$. It shows clearly that informed and uninformed investors have opposite price sensitivities to the fundamental shocks. $\phi_\Delta(t)$ measures the uninformed investors' sensitivity. It is smaller with imperfect and asymmetric information and largest right after the announcement. The uninformed investor becomes informed when information is perfect and symmetric. They would not "mistake" noise as fundamentals. As a result, their price sensitivities with respect to fundamentals becomes the largest at the announcements. On the other hand, the informed investors could benefit from their higher channel capacities on non-announcement days. They have more precise information about the fundamentals than the uninformed investors. Therefore, their price sensitivities with respect to fundamentals are larger. However, this information advantage is eliminated upon announcements. This explains the their smallest price sensitivities on announcement days. Although the price sensitivities are different for two types of investors, their summation, ϕ_y, is a constant. As shown in the next section, price responses to forecast errors are driven by time varying $\phi_\theta(t)$.

5.3 Quantitative Results

In order to match the empirical facts documented before, assume revisions occur at the middle of each quarter, when the professional forecast survey is submitted ($t = nT,\ \tau = 45$). Therefore, the stock market reaction to forecast revisions can be characterized as:

$$\begin{aligned} P_{t+\tau+1} - P_{t+\tau} &= [\phi(t+\tau+1) - \phi(t+\tau)] + \phi_D(D_{t+\tau+1} - D_{t+\tau}) \\ &+ \left[\phi_\theta(t+\tau+1)\tilde{\theta}_{t+\tau+1} - \phi_\theta(t+\tau)\tilde{\theta}_{t+\tau}\right] + \phi_y(\tilde{x}_{t+\tau+1} - \tilde{x}_{t+\tau}). \end{aligned}$$

Figure 5.1: Time Varying Price Sensitivities



To test whether stock market reactions to forecast revisions predict stock market reactions to forecast errors, run the following regression using model simulated data:

$$(A)\ P^{+}_{(n+1)T} - P^{-}_{(n+1)T} \quad = \quad \alpha_P + \beta_P \left(P_{nT+\tau+1} - P_{nT+\tau}\right) + \varepsilon_P.$$

In order to confirm the intuition that the predictable component must come from price reactions to noise, further decompose the price reactions to forecast errors into three parts (in terms of informed investor's private information set and public information) and perform the following experiments:

$$\begin{aligned}
(A.1) \quad & \phi_y \left[x_{(n+1)T} - \tilde{x}_{(n+1)T}\right] && = \alpha_P + \beta_{P2} \left(P_{nT+\tau+1} - P_{nT+\tau}\right) + \varepsilon_P \\
(A.2) \quad & \phi_\theta(0) \left[\theta_{(n+1)T} - \tilde{\theta}_{(n+1)T}\right] && = \alpha_P + \beta_{P3} \left(P_{nT+\tau+1} - P_{nT+\tau}\right) + \varepsilon_P \\
(A.3) \quad & \left[\phi_\theta(0) - \phi_\theta(T)\right] \tilde{\theta}_{(n+1)T} && = \alpha_P + \beta_{P4} \left(P_{nT+\tau+1} - P_{nT+\tau}\right) + \varepsilon_P.
\end{aligned}$$

Mapping into the empirical exercise, the relationship between consensus forecast revisions FR_t and consensus forecast errors FE_t can be expressed as

$$\begin{aligned}
FR_{nT} &= (1-\omega)\left[\hat{x}_{nT+\tau+1} - \hat{x}_{nT+\tau}\right] + \omega\left[\tilde{x}_{nT+\tau+1} - \tilde{x}_{nT+\tau}\right] \\
FE_{nT} &= x_{nT+\tau+1} - \left[(1-\omega)\,\hat{x}_{nT+\tau+1} + \omega\tilde{x}_{nT+\tau+1}\right].
\end{aligned}$$

To test whether consensus forecast revisions predict forecast errors, specify:

$$\begin{aligned}
(B)\ FE_{nT} &= \alpha_F + \beta_F FR_{nT} + \varepsilon_F. \\
(B.1)\ x_{(n+1)T+\tau} - \hat{x}_{(n+1)T+\tau} &= \alpha^1_{\hat{x}} + \beta^1_{\hat{x}}\left[\hat{x}_{(n+1)T+\tau} - \hat{x}_{nT+\tau+1}\right] + \varepsilon^1_{\hat{x}} \\
(B.2)\ x_{(n+1)T+\tau} - \tilde{x}_{(n+1)T+\tau} &= \alpha^1_{\tilde{x}} + \beta^1_{\tilde{x}}\left[\hat{x}_{(n+1)T+\tau} - \hat{x}_{nT+\tau+1}\right] + \varepsilon^1_{\tilde{x}}.
\end{aligned}$$

Table 5.2 summarizes the model implied quantitative results. The intuition can be summarized as follows. On non-announcement days, information is asymmetric. Since price contains information about fundamental cash-flows, investors try to extract information from it. A positive shock to

Table 5.2: Quantitative Results

	(A)	$(A.1)$	$(A.2)$	$(A.3)$	(B)	$(B.1)$	$(B.2)$
Coefficient	-0.1792***	-0.0000	0.0001	-0.2130***	0.4032***	0.0563	0.0950
	(0.006)	(0.005)	(0.001)	(0.003)	(0.055)	(0.039)	(0.067)
Constant	-0.000	-0.003**	-0.000	0.001	-0.000**	-0.000**	-0.000*
	(0.001)	(0.001)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)

This table presents the regression coefficients based on specifications in number (1) to (9). The simulation is 5e5 years with quarterly macroeconomic announcements and daily stock prices. The maximum ODE convergence tolerance is 2e-12 for the simulation. Panel A and B use the same paths of random shocks. Note: $^{*}p < 0.01;^{**} p < 0.005;^{***} p < 0.001$.

fundamental dividends and negative shock to noise θ_t both increase the stock price. Informed investors know exactly the fraction of shocks coming from θ_t because price is fully revealing to them. However, uninformed investors do not. Suppose there is a negative shock to θ_t. Informed investors rationally attribute the resulting increase in price as a combination of shocks from dividends and θ_t. This gives rise to an increase in their posterior beliefs about fundamentals $\tilde{x}_t$. As a result, price further increases to clear the market. This generates over-reactions to noise on non-announcement days. Upon announcements, uncertainty is reduced and information asymmetry is eliminated. The stock price must fall to correct for those mistaken accumulated beliefs about noise from uninformed investors on non-announcement days.

It is important to remember that because agents update and revise their beliefs through rational learning, the future estimation error in fundamentals can not be predicted ahead of time. Hence, only price reactions to noise, θ_t, can potentially be predicted by responses to revisions. The market price reacts to noise only if the equilibrium price upon announcements has different sensitivities to it (i.e. $\phi_\theta(t)$ is time-varying).

5.4 Robustness

Table 5.3 shows that the results are robust for alternative parameter values. In general, the results are robust in response to variation of the parameters. There are several things to be noticed. First, with a higher ω (e.g. $\omega = 0.7$), the proportion of uninformed investors increases, and the information asymmetry becomes more severe. In response, price becomes more sensitive to noise on non-announcement days. At the announcements, price reverses back by larger extent to its true value as information becomes symmetric. This explains the larger absolute value of β_P (0.182) with larger ω. Moreover, a larger population of uninformed investors naturally induces a more imperfect information in the economy. This increases the degree of information rigidity, which produces a higher β_F of 0.433. Second, higher channel capacity of the informed investor will increase the signal precision and decrease σ_s. As a result, the degree of information rigidity β_F is as high as 0.683 and price will be less sensitive to the noise with $|\beta_P|$ as small as 0.129 under the case of $\sigma_s = 0.2$.

Table 5.3: Parameter Robustness

	Benchmark	$\omega = 0.2$	$\omega = 0.7$	$\sigma_s = 0.2$	$\sigma_s = 1.5$	$\sigma_x = 0.4$	$\sigma_x = 0.7$	$\sigma_d = 0.8$	$\sigma_d = 1.2$
β_P	-0.1792	-0.1791	-0.1815	-0.1285	-0.2051	-0.0496	-0.6475	-0.2941	-0.1671
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.004)	(0.009)	(0.006)	(0.006)
β_F	0.4032	0.2024	0.4332	0.6826	0.1940	0.4359	0.3900	0.3081	0.4906
	(0.055)	(0.045)	(0.062)	(0.024)	(0.064)	(0.077)	(0.045)	(0.047)	(0.061)
		$\sigma_\theta = 1$	$\sigma_\theta = 5$	$a = 0.005$	$a = 0.015$	$b = 0.15$	$b = 0.25$	$\bar{x} = 1$	$\bar{x} = 6$
β_P		-0.0840	-0.2457	-0.4066	-0.1483	-0.5652	-0.0766	-0.2129	-0.2129
		(0.011)	(0.005)	(0.007)	(0.006)	(0.009)	(0.005)	(0.006)	(0.007)
β_F		0.3707	0.4080	0.4057	0.4057	0.4028	0.4027	0.4057	0.4057
		(0.054)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)

This table reports the robustness check for calibrated parameter values. "Benchmark" uses parameter values from Table 5.1. The maximum ODE convergence tolerance is 1e-10 for all simulations. All results are significant with $p < 0.001$.

5.5 Price Volatility

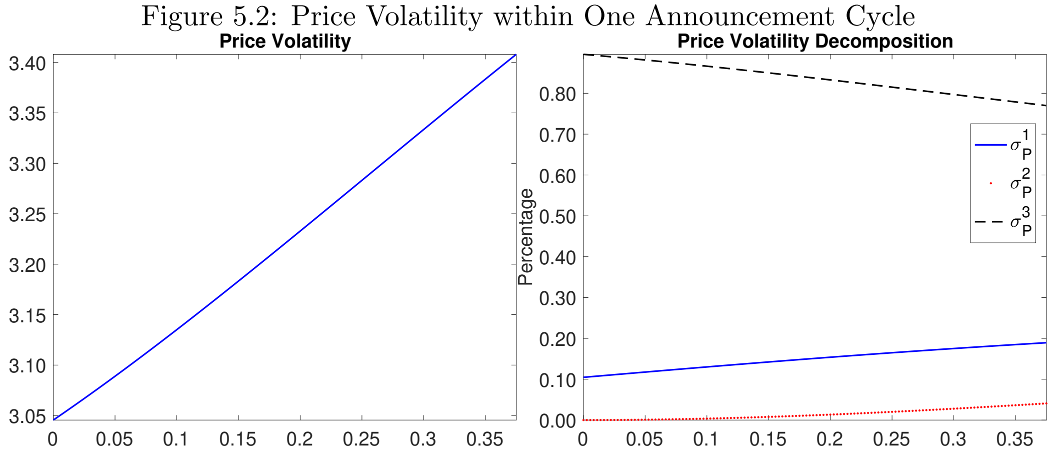
Price volatility takes the following form,

$$\sigma_P(t) = \sqrt{(\sigma_{P,1}(t))^2 + (\sigma_{P,1}(t))^2 + (\sigma_{P,3}(t))^2}\,.$$

where $\sigma_{P,1}(t) = \phi_D \sigma_D + \phi_y \frac{\hat{q}_t}{\sigma_D} - \phi_{\Delta,t}\left(\left(1 - h_{12,t}\phi_{x,t}\right)\frac{\hat{q}_t}{\sigma_D} - h_{11,t}\sigma_D\right)$, $\sigma_{P,2}(t) = \phi_y \frac{\hat{q}_t}{\sigma_s} - \phi_{\Delta,t}\left(1 - h_{12,t}\phi_{x,t}\right)\frac{\hat{q}_t}{\sigma_s}$, $\sigma_{P,3}(t) = \phi_{\theta,t}\sigma_\theta\left(1 + \phi_{\Delta,t}h_{12,t}\right)$.

Within each announcement cycle (one quarter), the price volatility and the decomposition of it are depicted in Figure 5.2. The left panel shows that volatility drops at the announcements ($t = 0$);

Figure 5.2: Price Volatility within One Announcement Cycle



and the right panel shows that $\sigma_{P,3}(t)$ is the driving force of the price volatility. This mainly comes from the sensitivity of price with respect to noise, i.e. $\phi_\theta(t)$, is smaller upon announcements relative to non-announcement days. Following announcements, both agents obtain full information. We show that a time-varying information structure can explain a seemingly puzzling empirical result, namely, that realized volatility does not change following announcements. Although increased information following announcements would by itself increase volatility, the endogenous reduction in price sensitivity offsets this.

6 Conclusion

This paper began by documenting a new empirical fact, namely, that stock market responses to macroeconomic forecast revisions negatively predict future market responses to forecast errors.

We then develop a model to show how this predictability arises in a Rational Expectations equilibrium. The key new ingredient in our model is periodic announcements of macroeconomic data, which enables investors to correct their existing estimation errors. Between announcements, estimation errors can accumulate, since the underlying state remains hidden. We then go on to show that a reasonably calibrated version of the model can not only replicate the sign of the relationship, it can also match the data *quantitatively.*

To simplify computation of the equilibrium, this paper relied on exponential preferences, which delivers a convenient linear pricing function (albeit with time-varying coefficients). Unfortunately, this specification makes the model ill-suited to study risk premia. In particular, using a generalized class of risk-sensitive preferences, Ai and Bansal (2018) show that macroeconomic announcements produce a significant 'announcement premium'. Hence, it might be useful to revisit the questions addressed here using their preferences. Another possible extension would be exploit other data sources on survey expectations. For example, rather than focus on the aggregate stock market using macroeconomic surveys, the analysis here could be replicated using I/B/E/S data on analyst forecasts of the earnings of individual firms. Finally, a third possible extension would be to examine the model's implications for trading volume. Our model predicts that trading volume should trend up between announcements, and then drop rapidly following an announcement. It would be interesting to see whether observed trading exhibits this sort of time variation.

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Appendix

6.1 Robustness

We check the robustness of this paper's main results in two perspectives: the measurement of forecast revision and the stock market reaction to it.

First, following CG15, define year-on-year annual forecast revision ($AFrev_t$) and forecast error ($AFerr_t$) as:

$$AFrev_t = \mathbb{F}_t x_{t+3,t} - \mathbb{F}_{t-1} x_{t+3,t} \quad (41)$$

$$AFerr_t = x_{t+3,t} - \mathbb{F}_t x_{t+3,t} \quad (42)$$

where $x_{t+3,t}$ denotes the average GDP growth rate over the current t and next three quarters (e.g. if $t = 0$, $x_{t+3,t} = \frac{x_0+x_1+x_2+x_3}{4}$). Annual forecast error refers to the annual average over differences between associated realized value and forecasts about t and next three quarters (all submitted at t). In this case, agents do a longer forecast for one year ahead instead of one quarter ahead. The results are shown in Panel A of Table 6.1.

Second, it is hard to determine on which days forecasters revise their beliefs, and the return may also reflects responses to cumulative revisions. One may argue that the revision has been started right after last submission day. Thus, to check the robustness, we use close-to-close return between current and last quarter submission deadline days, defining as: $QRrev_{nt} = \frac{close_{nt} - close_{(n-1)t}}{close_{(n-1)t}}$, where n denotes the integer index for a quarter. Similarly, as the survey is distributed to the forecasters after the advance estimate released by BEA at the end of each month (about two weeks before the submission deadline), we calculate $WWRrev_t$, the close-to-close return between survey deadline and last advance estimate of GDP releasing day.[28] In addition, we also check $WRrev_t$, defining as close-to-close returns between the survey deadline day and one week before it. The results are displayed in Panel B of Table 6.1.

Table 6.1 Panel A shows similar results as before. However, forecast revisions have stronger predictive power for forecast errors. It is easy to understand because annual forecast contains more belief revisions compared to the forecast only about the current quarter. Controlling for these new revision and error measurements, the results still hold, while R^2 increases from 18.4 to 19.6 percent (see column (5)).

From Panel B shows that the negative predictability remains significant for alternative measures of quarterly revision $QRrev_t$ and revision within one week $WRrev_t$, and the coefficients are almost the same around -0.03, even controlling for other variables.

[28]Close-to-open return between survey deadline and last advance estimate of GDP releasing day has similar results.

Table 6.1: Robustness Tests of GDP Growth Rate Expectations Formation in Stock Market

Panel A: measurement for forecast revision and forecast error						
	(1)	(2)	(3)	(4)	(5)	
	$AFerr_t$	$Rrev_t$		$Rann_t$		
$Rrev_t$					-0.186**	
					(0.087)	
$AFrev_t$	0.742***	0.911**		-0.287**	-0.151	
	(0.276)	(0.378)		(0.120)	(0.033)	
$AFerr_t$			0.031		0.092	
			(0.075)		(0.060)	
Δx_t					0.020	
					(0.021)	
Constant	0.149	0.141	0.121***	0.180**	0.160***	
	(0.162)	(0.108)	(0.059)	(0.060)	(0.055)	
N	188	84	97	104	77	
R^2	0.076	0.113	0.003	0.035	0.196	
Panel B: Measurement for stock market reaction to forecast revisions						
	(1)	(2)	(3)	(4)	(5)	(6)
$QRrev_t$	-0.028**				-0.031*	
	(0.014)				(0.018)	
$MRrev_t$		-0.014				
		(0.013)				
$WWRrev_t$			-0.046			
			(0.041)			
$WRrev_t$				-0.029**		-0.033*
				(0.014)		(0.018)
$Frev_t$					-0.015	-0.004
					(0.069)	(0.068)
$Ferr_t$					0.041	0.044
					(0.063)	(0.064)
Δx_t					0.025	0.025
					(0.038)	(0.038)
Constant	0.172**	0.186**	0.164**	0.181**	0.170*	0.183**
	(0.072)	(0.084)	(0.069)	(0.072)	(0.087)	(0.087)
N	73	69	83	74	72	73
R^2	0.100	0.028	0.043	0.102	0.122	0.125

This table reports coefficient estimates of regressing first row variables on first column variables using OLS following CG15. Newey-West (lagged 5) standard errors are in parentheses.

6.2 Proof for the Two-Period Model

The risky asset demand for agent i is: $\alpha_i = \rho_x \bar{x} + \rho_\varepsilon (x + \varepsilon_i) + \rho_e (x + e) - (\rho_x + \rho_\varepsilon + \rho_e) P$. Integrate the demand of all agents and using the market clearing condition immediately gives:

$$\rho_x \bar{x} + \rho_\varepsilon (x - \theta/\rho_\varepsilon) + \rho_e (x + e) = (\rho_x + \rho_\varepsilon + \rho_e) P.$$

Because both sides must be $x+e$ measurable, this gives: $x-\theta/\rho_\varepsilon = x+e$ or $e=\theta/\rho_\varepsilon$. Therefore, the equilibrium price in equation (8) could be obtained.

Note that $\mathrm{Var}(e) = \mathrm{Var}(\theta/\rho_\varepsilon) = \frac{1}{\rho_\varepsilon^2 \rho_\theta}$. This gives $\rho_e = \rho_\varepsilon^2 \rho_\theta$.

6.3 Proof for Uninformed Investors' Filtering Problem

The uninformed investor's learning problem is that he/she tries to learn $\vartheta_t = \begin{bmatrix} x_t & \hat{x}_t & \theta_t \end{bmatrix}^\top$, by observing $\xi_t = \begin{bmatrix} D_t & \zeta_t \end{bmatrix}^\top$.

Solving for stochastic process of ζ_t gives

$$\begin{aligned} d\zeta_t &= \left[b\bar{x}\phi_x(t) + \left((a-b)\phi_x(t) + \phi_x'(t)\right)\hat{x}_t + \phi_\theta'(t)\theta_t - a\zeta\right]dt \\ &\quad + \frac{\hat{q}(t)}{\sigma_D}\phi_x(t)\,d\hat{B}_{D,t} + \frac{\hat{q}(t)}{\sigma_s}\phi_x(t)\,d\hat{B}_{s,t} + \sigma_\theta\phi_\theta(t)\,dB_{\theta,t}, \end{aligned} \tag{43}$$

and

$$dD_t = (\hat{x}_t - \kappa D_t)\,dt + \sigma_D d\hat{B}_{D,t}. \tag{44}$$

Rewrite the $d\zeta$ and $d\hat{x}_t$ in terms of the fundamental innovations:

$$\begin{aligned} d\zeta_t &= \left[b\bar{x}\phi_x(t) + \left((a-b)\phi_x(t) + \phi_x'(t)\right)\hat{x}_t + \phi_\theta'(t)\theta_t - a\zeta + \hat{\sigma}\phi_x(t)\hat{q}(t)(x_t - \hat{x}_t)\right]dt \\ &\quad + \frac{\hat{q}(t)}{\sigma_D}\phi_x(t)\,dB_{D,t} + \frac{\hat{q}(t)}{\sigma_s}\phi_x(t)\,dB_{s,t} + \sigma_\theta\phi_\theta(t)\,dB_{\theta,t}, \\ d\hat{x}_t &= \left[b(\bar{x} - \hat{x}_t) + \hat{\sigma}\hat{q}(t)(x_t - \hat{x}_t)\right]dt + \frac{\hat{q}(t)}{\sigma_D}dB_{D,t} + \frac{\hat{q}(t)}{\sigma_s}dB_{s,t}. \end{aligned}$$

Denote the independent Brownian motions $W_1 = \begin{bmatrix} B_{x,t} & 0 & B_{\theta,t} \end{bmatrix}^\top$, $W_2 = \begin{bmatrix} B_{D,t} & B_{s,t} \end{bmatrix}^\top$. Consider the 3 plus 2 dimensional Gaussian random process (ϑ_t, ξ_t), $t \in [(n-1)T^+, nT^-]$, $n = 1, 2, \ldots$, with

$$\begin{aligned} d\vartheta_t &= \left[a_0(t) + a_1(t)\vartheta_t + a_2(t)\xi_t\right]dt + \sum_{i=1}^{2} b_i(t)\,dW_i(t), \\ d\xi_t &= \left[A_0(t) + A_1(t)\vartheta_t + A_2(t)\xi_t\right]dt + \sum_{i=1}^{2} B_i(t)\,dW_i(t). \end{aligned}$$

where

$$a_0(t) = \begin{bmatrix} b\bar{x} \\ b\bar{x} \\ 0 \end{bmatrix},\ a_1(t) = \begin{bmatrix} -b & 0 & 0 \\ \hat{\sigma}\hat{q}_t & -b-\hat{\sigma}\hat{q}_t & 0 \\ 0 & 0 & -a \end{bmatrix},\ a_2(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$b_1(t) = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_\theta \end{bmatrix}, b_2(t) = \begin{bmatrix} 0 & 0 \\ \frac{\hat{q}_t}{\sigma_D} & \frac{\hat{q}_t}{\sigma_s} \\ 0 & 0 \end{bmatrix};$$

$$A_0(t)=\begin{bmatrix}0\\ b\bar{x}\phi_x(t)\end{bmatrix},\ A_1(t)=\begin{bmatrix}1 & 0 & 0\\ \hat{\sigma}\phi_{x,t}\hat{q}_t & (a-b)\phi_{x,t}+\phi'_{x,t}-\hat{\sigma}\phi_{x,t}\hat{q}_t & \phi'_{\theta,t}\end{bmatrix},\ A_2(t)=\begin{bmatrix}-\kappa & 0\\ 0 & -a\end{bmatrix},$$

$$B_1(t)=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \sigma_\theta\phi_{\theta,t}\end{pmatrix},\ B_2(t)=\begin{pmatrix}\sigma_D & 0\\ \frac{\hat{q}_t}{\sigma_D}\phi_{x,t} & \frac{\hat{q}_t}{\sigma_s}\phi_{x,t}\end{pmatrix}.$$

This further gives

$$(b\circ b)(t)\equiv b_1(t)\,b_1^\top(t)+b_2(t)\,b_2^\top(t)=\begin{bmatrix}\sigma_x^2 & 0 & 0\\ 0 & \hat{\sigma}\hat{q}_t^2 & 0\\ 0 & 0 & \sigma_\theta^2\end{bmatrix},$$

$$(b\circ B)(t)\equiv b_1(t)\,B_1^\top(t)+b_2(t)\,B_2^\top(t)=\begin{bmatrix}0 & 0\\ \hat{q}_t & \hat{\sigma}\phi_{x,t}\hat{q}_t^2\\ 0 & \sigma_\theta^2\phi_{\theta,t}\end{bmatrix},$$

$$(B\circ B)(t)\equiv B_1(t)\,B_1^\top(t)+B_2(t)\,B_2^\top(t)=\begin{bmatrix}\sigma_D^2 & \phi_{x,t}\hat{q}_t\\ \phi_{x,t}\hat{q}_t & \hat{\sigma}\phi_{x,t}^2\hat{q}_t^2+\sigma_\theta^2\phi_{\theta,t}^2\end{bmatrix}.$$

Applying Theorem 10.3 from Liptser and Shiryaev (2001), the solutions of the system of equations can be characterized as:

$$\begin{aligned}
d\tilde{\vartheta}_t &= \left[a_0(t)+a_1(t)\vartheta_t+a_2(t)\xi_t\right]dt+\left[(b\circ B)(t)+\tilde{q}_tA_1^\top(t)\right](B\circ B)(t)^{-1}\\
&\quad\times\left[d\xi_t-\left(A_0(t)+A_1(t)\tilde{\vartheta}_t+A_2(t)\xi_t\right)dt\right],\\
d\tilde{q}_t &= \left[a_1(t)\tilde{q}_t+\tilde{q}_ta_1^\top(t)+(b\circ b)(t)-\left((b\circ B)(t)+\tilde{q}_tA_1^\top(t)\right)(B\circ B)(t)^{-1}\left((b\circ B)(t)+\tilde{q}_tA_1^\top(t)\right)^\top\right]dt
\end{aligned}$$

where $\tilde{q}(t)=\begin{bmatrix}\tilde{q}_{11}(t) & \tilde{q}_{12}(t) & \tilde{q}_{13}(t)\\ * & \tilde{q}_{22}(t) & \tilde{q}_{23}(t)\\ * & * & \tilde{q}_{33}(t)\end{bmatrix}$ is the positive definite symmetric variance-covariance matrix.

Putting parameter matrices back to above equations gives the filtering equations in the main text, with the coefficients and the vector of innovation processes defined as follows:

$$h_{11}(t)=\frac{\tilde{q}_{22,t}\left[\hat{q}_t^2\phi_x^2\left(1+\frac{\sigma_s^2}{\sigma_D^2}\right)-\hat{q}_t\sigma_s^2\phi_x^2\left(a-b+\frac{\phi'_x}{\phi_x}-\frac{\phi'_\theta}{\phi_\theta}\right)+\sigma_\theta^2\phi_\theta^2\sigma_s^2\right]+\hat{q}_t\sigma_\theta^2\phi_\theta^2\sigma_s^2}{\sigma_D^2\left(\hat{q}_t^2\phi_x^2+\sigma_\theta^2\phi_\theta^2\sigma_s^2\right)},\ h_{21}(t)=-\frac{\phi_{x,t}}{\phi_{\theta,t}}h_{11}(t),$$

$$h_{12}(t)=\frac{\tilde{q}_{22,t}(a-b+\frac{\phi'_x}{\phi_x}-\frac{\phi'_\theta}{\phi_\theta}-\frac{\hat{q}_t}{\sigma_D^2})\sigma_s^2\phi_x+\hat{q}_t^2\phi_x}{\hat{q}_t^2\phi_x^2+\sigma_\theta^2\phi_\theta^2\sigma_s^2},\ \text{and}\ h_{22}(t)=\frac{1-\phi_{x,t}h_{12}(t)}{\phi_{\theta,t}}; \tag{45}$$

and

$$\begin{aligned}
d\tilde{B}_{D,t} &= dD_t - (\tilde{x}_t - \kappa D_t)\, dt, \\
d\tilde{B}_{\zeta,t} &= d\zeta_t - \left[b\bar{x}\phi_x(t) + \big((a-b)\phi_x(t) + \phi_x'(t)\big)\tilde{x}_t + \phi_\theta'(t)\tilde{\theta}_t - a\zeta_t\right] dt. \qquad (46)
\end{aligned}$$

Note that from Equation (14), $\theta_t = \frac{1}{\phi_{\theta,t}}\left(\zeta_t - \phi_{x,t}\hat{x}_t\right)$ and $\tilde{\theta}_t = \frac{1}{\phi_{\theta,t}}\left(\zeta_t - \phi_{x,t}\tilde{x}_t\right)$. Applying law of total covariance,

$$\begin{aligned}
\tilde{q}_{11,t} &= \mathrm{Var}\left(x_t \mid \mathcal{F}^u\right) = \mathbb{E}\left[\mathrm{Var}\left(x_t \mid \mathcal{F}^i\right) \mid \mathcal{F}^u\right] + \mathrm{Var}\left(\mathbb{E}\left[x_t \mid \mathcal{F}^i\right] \mid \mathcal{F}^u\right) = \hat{q}_t + \tilde{q}_{22,t}, \\
\tilde{q}_{12,t} &= \mathrm{Cov}\left(x_t, \hat{x}_t \mid \mathcal{F}^u\right) = \mathbb{E}\left[\mathrm{Cov}\left(x_t, \hat{x}_t \mid \mathcal{F}^i\right) \mid \mathcal{F}^u\right] + \mathrm{Cov}\left(\mathbb{E}\left[x_t \mid \mathcal{F}^i\right], \mathbb{E}\left[\hat{x}_t \mid \mathcal{F}^i\right] \mid \mathcal{F}^u\right) = \tilde{q}_{22,t}, \\
\tilde{q}_{13,t} &= \mathrm{Cov}\left(x_t, \theta_t \mid \mathcal{F}^u\right) = \mathrm{Cov}\left[x_t, \frac{1}{\phi_{\theta,t}}\left(\zeta_t - \phi_{x,t}\hat{x}_t\right) \mid \mathcal{F}^u\right] = -\frac{\phi_{x,t}}{\phi_{\theta,t}}\tilde{q}_{12,t}. \\
\tilde{q}_{23,t} &= \mathrm{Cov}\left(\hat{x}_t, \theta_t \mid \mathcal{F}^u\right) = -\frac{\phi_{x,t}}{\phi_{\theta,t}}\tilde{q}_{22,t}, \\
\tilde{q}_{33,t} &= \mathrm{Var}\left(\theta_t \mid \mathcal{F}^u\right) = \frac{\phi_{x,t}^2}{\phi_{\theta,t}^2}\tilde{q}_{22,t}.
\end{aligned}$$

Therefore, the following equivalent conditions could be easily derived:

$$\begin{aligned}
\tilde{q}_{11,t} &= \hat{q}_t + \tilde{q}_{22,t},\ \tilde{q}_{33,t} = \frac{\phi_{x,t}^2}{\phi_{\theta,t}^2}\tilde{q}_{22,t}, \qquad (47) \\
\tilde{q}_{12,t} &= \tilde{q}_{22,t},\ \tilde{q}_{23,t} = -\frac{\phi_{x,t}}{\phi_{\theta,t}}\tilde{q}_{22,t},\ \tilde{q}_{13,t} = -\frac{\phi_{x,t}}{\phi_{\theta,t}}\tilde{q}_{22,t}, \qquad (48)
\end{aligned}$$

where $\tilde{q}_{22,t}$ satisfies the following stochastic process:

$$\begin{aligned}
d\tilde{q}_{22,t} = \; & \frac{1}{\hat{q}_t^2\phi_x^2 + \sigma_\theta^2\phi_\theta^2\sigma_s^2}\Bigg\{-\tilde{q}_{22,t}^2\Bigg[\hat{q}_t^2\left(\frac{\phi_x}{\sigma_D}\right)^2\left(1 + \frac{\sigma_s^2}{\sigma_D^2}\right) - 2\hat{q}_t\left(\frac{\sigma_s\phi_x}{\sigma_D}\right)^2\left(a - b + \frac{\phi_x'}{\phi_x} - \frac{\phi_\theta'}{\phi_\theta}\right) \\
& + \sigma_s^2\phi_x^2\left(a - b + \frac{\phi_x'}{\phi_x} - \frac{\phi_\theta'}{\phi_\theta}\right)^2 + \left(\sigma_\theta\phi_\theta\frac{\sigma_s}{\sigma_D}\right)^2\Bigg] \\
& - 2\tilde{q}_{22,t}\left[\hat{q}_t^2\phi_x^2\left(a + \frac{\phi_x'}{\phi_x} - \frac{\phi_\theta'}{\phi_\theta}\right) + \hat{q}_t\left(\sigma_\theta\phi_\theta\frac{\sigma_s}{\sigma_D}\right)^2 + b\left(\sigma_\theta\phi_\theta\sigma_s\right)^2\right] + \left(\hat{q}_t\sigma_\theta\phi_\theta\right)^2\Bigg\} dt. \qquad (49)
\end{aligned}$$

6.4 Proof for Stochastic Process of Estimation Difference

The stochastic process for estimation difference $\Delta \equiv \hat{x}_t - \tilde{x}_t$ between two types of investors could be derived directly from Equations (17) and (19):

$$\begin{aligned}
d\Delta_t \equiv d\hat{x}_t - d\tilde{x}_t = \; & -b\Delta_t + \frac{\hat{q}_t}{\sigma_D}d\hat{B}_{D,t} + \frac{\hat{q}_t}{\sigma_s}d\hat{B}_{s,t} - h_{11,t}\left[dD_t - (\tilde{x}_t - \kappa D_t)\, dt\right] \\
& - h_{12,t}\left[d\zeta_t - \left[b\bar{x}\phi_{x,t} + \left((a-b)\phi_{x,t} + \phi_{x,t}'\right)\tilde{x}_t + \phi_{\theta,t}'\tilde{\theta}_t - a\zeta_t\right] dt\right].
\end{aligned}$$

Substituting Equations (43) and (44) would give

$$
\begin{aligned}
d\Delta_t &= -\left[b + h_{11,t} + \left(\left(a - b - \frac{\phi'_{\theta,t}}{\phi_{\theta,t}}\right)\phi_{x,t} + \phi'_{x,t}\right)h_{12,t}\right]\Delta_t dt \\
&\quad + \left[\left(1 - h_{12,t}\phi_{x,t}\right)\frac{\hat{q}_t}{\sigma_D} - h_{11,t}\sigma_D\right]d\hat{B}_{D,t} + \left(1 - h_{12,t}\phi_{x,t}\right)\frac{\hat{q}_t}{\sigma_s}d\hat{B}_{s,t} - h_{12,t}\sigma_\theta\phi_{\theta,t}dB_{\theta,t}
\end{aligned}
\tag{50}
$$

where $a_\Delta(t) = b + h_{11,t} + \left[\left(a - b - \frac{\phi'_{\theta,t}}{\phi_{\theta,t}}\right)\phi_{x,t} + \phi'_{x,t}\right]h_{12,t}$, $\sigma_{\Delta D}(t) = \left(1 - h_{12,t}\phi_{x,t}\right)\frac{\hat{q}_t}{\sigma_D} - h_{11,t}\sigma_D$, $\sigma_{\Delta s}(t) = \left(1 - h_{12,t}\phi_{x,t}\right)\frac{\hat{q}_t}{\sigma_s}$, and $\sigma_{\Delta\theta} = -h_{12,t}\sigma_\theta\phi_{\theta,t}$.

6.5 Solving for Excess Returns

The instantaneous excess return satisfies $dQ_t = dP_t + D_t dt - rP_t dt$.

First, under informed investor's information set, substituting Expressions (11), (15), (17), (21) and (44) yields

$$
dQ_t^i = \{e_0(t) + [1 - (\kappa + r)\phi_D(t)]D_t + e_\theta(t)\theta_t + e_\Delta(t)\Delta_t\}dt + b_D^i(t)d\hat{B}_{D,t} + b_s^i(t)d\hat{B}_{s,t} + b_\theta^i(t)dB_{\theta,t}.
$$

From market clearing conditions, the coefficients of D_t must be 0. Hence,

$$
\phi_D = \frac{1}{\kappa + r}.
$$

Therefore, under informed investor's information set $\mathcal{F}^i$, the excess return Q_t^i satisfies:

$$
dQ_t^i = [e_0(t) + e_\theta(t)\theta_t + e_\Delta(t)\Delta_t]dt + b_D^i(t)d\hat{B}_{D,t} + b_s^i(t)d\hat{B}_{s,t} + b_\theta^i(t)dB_{\theta,t}, \tag{51}
$$

where

$$
\begin{aligned}
e_0(t) &= \phi'(t) - r\phi(t) + b\bar{x}\phi_y(t), \\
e_\theta(t) &= \phi'_\theta(t) - (a + r)\phi_\theta(t), \\
e_\Delta(t) &= -\phi'_\Delta(t) + \phi_\Delta(t)\left[b + r + h_{11}(t) + \phi_x(t)\left(a - b + \frac{\phi'_x(t)}{\phi_x(t)} - \frac{\phi'_\theta(t)}{\phi_\theta(t)}\right)h_{12}(t)\right], \\
b_D^i(t) &= \phi_x(t)[1 + \phi_\Delta(t)h_{12}(t)]\frac{\hat{q}(t)}{\sigma_D} + \phi_\Delta(t)h_{11}(t)\sigma_D + \phi_D\sigma_D, \\
b_s^i(t) &= [1 + \phi_\Delta(t)h_{12}(t)]\frac{\hat{q}(t)}{\sigma_s}, \\
b_\theta^i(t) &= [1 + \phi_\Delta(t)h_{12}(t)]\sigma_\theta\phi_\theta(t).
\end{aligned}
$$

Second, for uninformed investor, rewrite dQ_t^i into dQ_t^u by guessing

$$
dQ_t^u = \left[e_0(t) + e_\theta(t)\tilde{\theta}_t + e_x(t)\tilde{x}_t\right]dt + b_D^u(t)d\tilde{B}_{D,t} + b_\zeta^u(t)d\tilde{B}_{\zeta,t}.
$$

Using Equation (14) and $dQ_t^i - dQ_t^u = 0$ yields:

$$\left[-e_\theta(t)\frac{\phi_x(t)}{\phi_\theta(t)}+e_\Delta(t)+e_x(t)\right]\Delta_t dt+b_D^i(t)\,d\hat{B}_{D,t}+b_s^i(t)\,d\hat{B}_{s,t}+b_\theta^i(t)\,dB_{\theta,t}-b_D^u(t)\left[\Delta_t dt+\sigma_D d\hat{B}_{D,t}\right]$$

$$-b_\varsigma^u(t)\left[\left(\left(a-b-\frac{\phi_\theta'(t)}{\phi_\theta(t)}\right)\phi_x(t)+\phi_x'(t)\right)\Delta_t dt+\frac{\hat{q}(t)}{\sigma_D}\phi_x(t)\,d\hat{B}_{D,t}+\frac{\hat{q}(t)}{\sigma_s}\phi_x(t)\,d\hat{B}_{s,t}+\sigma_\theta\phi_\theta(t)\,dB_{\theta,t}\right]=0.$$

Hence,

$$\begin{aligned} b_D^u(t) &= \phi_D+h_{11}(t)\,\phi_\Delta(t) && (52)\\ b_\varsigma^u(t) &= 1+h_{12}(t)\,\phi_\Delta(t). && (53)\end{aligned}$$

Assuming value functions do not depend on $\hat{x}_t$ (see Appendix 6.9 for the proof) would give $e_x(t)=0$. This implies that $\phi_y(t)$ is a constant:

$$\phi_y=\phi_x(t)+\phi_\Delta(t)=\frac{\phi_D}{b+r}.$$

6.6 Solving for Optimization Problems in the Interior

Solving the Informed Investor's Optimization Problem. Conjecture the informed investor's value function takes the form of $J\left(t,W^i,\theta,\Delta\right)=-e^{-rW^i-g(t,\theta,\Delta)}$, where $g(t,\theta,\Delta)=g(t)+g_\theta(t)\,\theta_t+\frac{1}{2}g_{\theta\theta}(t)\,\theta_t^2+g_\Delta(t)\,\Delta_t+\frac{1}{2}g_{\Delta\Delta}(t)\,\Delta_t^2$. Using Ito's Lemma, the HJB equation is:[29]

$$\begin{aligned}\rho J &= -e^{-C^i}+J_t+J_W\left[rW^i-C^i+\alpha\left(e_0(t)+e_\theta(t)\,\theta+e_\Delta(t)\,\Delta\right)\right]\\ &\quad+\frac{1}{2}J_{WW}\alpha^2\left(\sigma_Q^i(t)\right)^2+\alpha J_{W\theta}\sigma_\theta b_\theta^i(t)+\alpha J_{W\Delta}\left(\sigma_{Q\Delta}^i(t)\right)^2-J_\theta a\theta\\ &\quad-J_\Delta a_\Delta(t)\,\Delta+\frac{1}{2}J_{\theta\theta}\sigma_\theta^2+\frac{1}{2}J_{\Delta\Delta}\left(\sigma_\Delta(t)\right)^2+J_{\theta\Delta}\sigma_\theta\sigma_{\Delta\theta}(t),\end{aligned}$$

where

$$\begin{aligned}\left(\sigma_Q^i(t)\right)^2 &= \left(b_D^i(t)\right)^2+\left(b_s^i(t)\right)^2+\left(b_\theta^i(t)\right)^2,\\ \left(\sigma_{Q\Delta}^i(t)\right)^2 &= b_D^i(t)\,\sigma_{\Delta D}(t)+b_s^i(t)\,\sigma_{\Delta s}(t)+b_\theta^i(t)\,\sigma_{\Delta\theta}(t), && (54)\\ \left(\sigma_\Delta(t)\right)^2 &= \sigma_{\Delta D}^2(t)+\sigma_{\Delta s}^2(t)+\sigma_{\Delta\theta}^2(t).\end{aligned}$$

Under the guessed value function form, $J_t=-\frac{\partial g}{\partial t}J$, $J_W=-rJ$, $J_{WW}=r^2J$, $J_\theta=-\frac{\partial g}{\partial\theta}J$, $J_\Delta=-\frac{\partial g}{\partial\Delta}J$, $J_{\theta\theta}=\left[\left(\frac{\partial g}{\partial\theta}\right)^2-\frac{\partial^2 g}{\partial\theta^2}\right]J$, $J_{\Delta\Delta}=\left[\left(\frac{\partial g}{\partial\Delta}\right)^2-\frac{\partial^2 g}{\partial\Delta^2}\right]J$, $J_{W\theta}=r\frac{\partial g}{\partial\theta}J$, $J_{W\Delta}=r\frac{\partial g}{\partial\Delta}J$, $J_{\Delta\theta}=\left(\frac{\partial g}{\partial\theta}\frac{\partial g}{\partial\Delta}-\frac{\partial^2 g}{\partial\theta\partial\Delta}\right)J$.

The FOC with respect to C^i is: $rW^i-C^i=\ln r-g(t,\theta,\Delta)$. Substituting these expressions

[29] For simplicity, we drop the unnecessary time scripts.

into this FOC, HJB can be rewritten as

$$
\begin{aligned}
0 &= r-\rho-\frac{\partial g}{\partial t}-r\left[\ln r-g+\alpha\left(e_0(t)+e_\theta(t)\theta+e_\Delta(t)\Delta\right)\right]+\frac{1}{2}r^2\alpha^2\left(\sigma_Q^i(t)\right)^2+\alpha r\frac{\partial g}{\partial\theta}\sigma_\theta b_\theta^i(t)\\
&+\alpha r\frac{\partial g}{\partial\Delta}\left(\sigma_{Q\Delta}^i(t)\right)^2+\frac{\partial g}{\partial\theta}a\theta+\frac{\partial g}{\partial\Delta}a_\Delta(t)\Delta+\frac{1}{2}\left[\left(\frac{\partial g}{\partial\theta}\right)^2-\frac{\partial^2 g}{\partial\theta^2}\right]\sigma_\theta^2\\
&+\frac{1}{2}\left[\left(\frac{\partial g}{\partial\Delta}\right)^2-\frac{\partial^2 g}{\partial\Delta^2}\right]\left(\sigma_\Delta(t)\right)^2+\left(\frac{\partial g}{\partial\theta}\frac{\partial g}{\partial\Delta}-\frac{\partial^2 g}{\partial\theta\partial\Delta}\right)\sigma_\theta\sigma_{\Delta\theta}(t).
\end{aligned}
$$

Moreover, FOC with respect to α gives $\alpha=\frac{e_0(t)+e_\theta(t)\theta+e_\Delta(t)\Delta-\frac{\partial g}{\partial\theta}b_\theta^i(t)\sigma_\theta-\frac{\partial g}{\partial\Delta}\left(\sigma_{Q\Delta}^i(t)\right)^2}{r\left(\sigma_Q^i(t)\right)^2}$. Under the guessed quadratic form of $g(t,\theta,\Delta)$, the derivatives, substituting expressions in (54) into the above equation yields

$$\alpha_t=\alpha_0(t)+\alpha_\theta(t)\theta_t+\alpha_\Delta(t)\Delta_t,$$

where $\alpha_0(t)=\frac{e_0(t)-b_\theta^i(t)\sigma_\theta g_\theta(t)-\left(\sigma_{Q\Delta}^i(t)\right)^2 g_\Delta(t)}{r\left(\sigma_Q^i(t)\right)^2}$, $\alpha_\theta(t)=\frac{e_\theta(t)-b_\theta^i(t)\sigma_\theta g_{\theta\theta}(t)-\left(\sigma_{Q\Delta}^i(t)\right)^2 g_{\theta\Delta}(t)}{r\left(\sigma_Q^i(t)\right)^2}$, and $\alpha_\Delta(t)=\frac{e_\Delta(t)-b_\theta^i(t)\sigma_\theta g_{\theta\Delta}(t)-\left(\sigma_{Q\Delta}^i(t)\right)^2 g_{\Delta\Delta}(t)}{r\left(\sigma_Q^i(t)\right)^2}$.

Finally, substituting the optimal policies back into the HJB equation and matching coefficients of the value function would give the following ODEs system of informed investor's value function coefficients.

Solving the Uninformed Investor's Optimization Problem. Similarly, the HJB equation of the uninformed investor's problem is:

$$
\begin{aligned}
\rho V &= -e^{-C^u}+V_t+V_W\left[rW^u-C^u+\beta\left(e_0(t)+e_\theta(t)\tilde{\theta}\right)\right]+\frac{1}{2}V_{WW}\beta^2\left(\sigma_Q^u(t)\right)^2\\
&+\beta V_{W\theta}\left(\sigma_{Q\theta}^u(t)\right)^2-V_\theta a\tilde{\theta}+\frac{1}{2}V_{\theta\theta}\left(\sigma_\theta^u\right)^2
\end{aligned}
$$

where

$$
\begin{aligned}
\left(\sigma_Q^u(t)\right)^2 &= \left(b_D^u(t)\right)^2+\left(b_\zeta^u(t)\right)^2\\
\left(\sigma_{Q\theta}^u(t)\right)^2 &= h_{21}(t)b_D^u(t)+h_{22}(t)b_\zeta^u(t)\\
\left(\sigma_\theta^u\right)^2 &= h_{21}^2(t)+h_{22}^2(t).
\end{aligned}
\tag{55}
$$

Furthermore, conjecture the uninformed investor's value function would be of the form: $V\left(t,W^u,\tilde{\theta}\right)=-e^{-rW^u-f(t,\tilde{\theta})}$, where $f\left(t,\tilde{\theta}\right)=f(t)+f_\theta(t)\tilde{\theta}_t+\frac{1}{2}f_{\theta\theta}(t)\tilde{\theta}_t^2$. FOC with respect to C^u is: $rW^u-C^u=$

$\ln r - f\left(t, \tilde{\theta}\right)$. Substituting this into HJB gives:

$$\begin{aligned}
\rho - r &= -\frac{\partial f}{\partial t} - r\left[\ln r - f + \beta\left(e_0(t) + e_\theta(t)\tilde{\theta}\right)\right] + \frac{1}{2}r^2\beta^2\left(\sigma_Q^u(t)\right)^2 + \beta r\frac{\partial f}{\partial \tilde{\theta}}\left(\sigma_{Q\theta}^u(t)\right)^2 \\
&\quad + \frac{\partial f}{\partial \tilde{\theta}}a\tilde{\theta} + \frac{1}{2}\left[\left(\frac{\partial f}{\partial \tilde{\theta}}\right)^2 - \frac{\partial^2 f}{\partial \tilde{\theta}^2}\right]\left(\sigma_\theta^u\right)^2.
\end{aligned}$$

Under the guessed form of $f\left(t, \tilde{\theta}\right)$, the derivatives could be expressed explicitly as: $\frac{\partial f}{\partial t} = f'(t) + f'_\theta(t)\tilde{\theta}_t + \frac{1}{2}f'_{\theta\theta}(t)\tilde{\theta}_t^2$, $\frac{\partial f}{\partial \tilde{\theta}} = f_\theta(t) + f_{\theta\theta}(t)\tilde{\theta}_t$, $\frac{\partial^2 f}{\partial \tilde{\theta}^2} = f_{\theta\theta}(t)$. Moreover, FOC with respect to β gives $\beta_t = \frac{e_0(t) + e_\theta(t)\tilde{\theta} - \frac{\partial f}{\partial \tilde{\theta}}\left(\sigma_{Q\theta}^u(t)\right)^2}{r\left(\sigma_Q^u(t)\right)^2}$. Substituting expressions in (55), and (23) into the above equation yields the optimal risk asset demand for uninformed investor as

$$\beta_t = \beta_0(t) + \beta_\theta(t)\tilde{\theta}_t,$$

where $\beta_0(t) = \frac{e_0(t) - \left(\sigma_{Q\theta}^u(t)\right)^2 f_\theta(t)}{r\left(\sigma_Q^u(t)\right)^2}$, and $\beta_\theta(t) = \frac{e_\theta(t) - \left(\sigma_{Q\theta}^u(t)\right)^2 f_{\theta\theta}(t)}{r\left(\sigma_Q^u(t)\right)^2}$.

Finally, substituting the optimal policy functions back into the HJB equation and matching coefficients of the value function would give the following ODEs system of uninformed investor's value function coefficients.

Solving for Market Clearing Conditions The market clear conditions expressed in Equation (33), (34), and (35), together with the coefficients system yield the following ODEs system of coefficients:

$$\begin{aligned}
g'(t) &= r - \rho - r\ln r + rg(t) - \frac{1}{2}r^2\left(\sigma_Q^i(t)\right)^2\alpha_0^2(t) + \frac{1}{2}\sigma_\theta^2\left[g_\theta^2(t) - g_{\theta\theta}(t)\right] \\
&\quad + \frac{1}{2}\sigma_\Delta^2(t)\left[g_\Delta^2(t) - g_{\Delta\Delta}(t)\right] + \sigma_\theta\sigma_{\Delta\theta}(t)\left[g_\theta(t)g_\Delta(t) - g_{\theta\Delta}(t)\right], \\
g'_\theta(t) &= rg_\theta(t) - r^2\left(\sigma_Q^i(t)\right)^2\alpha_0(t)\alpha_\theta(t) + ag_\theta(t) + \sigma_\theta^2 g_\theta(t)g_{\theta\theta}(t) \\
&\quad + \sigma_\Delta^2(t)g_\Delta(t)g_{\theta\Delta}(t) + \sigma_\theta\sigma_{\Delta\theta}(t)\left[g_\theta(t)g_{\theta\Delta}(t) + g_{\theta\theta}(t)g_\Delta(t)\right], \\
g'_\Delta(t) &= rg_\Delta(t) - r^2\left(\sigma_Q^i(t)\right)^2\alpha_0(t)\alpha_\Delta(t) + a_\Delta(t)g_\Delta(t) + \sigma_\theta^2 g_\theta(t)g_{\theta\Delta}(t) \\
&\quad + \sigma_\Delta^2(t)g_\Delta(t)g_{\Delta\Delta}(t) + \sigma_\theta\sigma_{\Delta\theta}(t)\left[g_\theta(t)g_{\Delta\Delta}(t) + g_{\theta\Delta}(t)g_\Delta(t)\right], \\
g'_{\theta\theta}(t) &= rg_{\theta\theta}(t) - r^2\left(\sigma_Q^i(t)\right)^2\alpha_\theta^2(t) \\
&\quad + 2ag_{\theta\theta}(t) + \sigma_\theta^2 g_{\theta\theta}^2(t) + \sigma_\Delta^2(t)g_{\theta\Delta}^2(t) + 2\sigma_\theta\sigma_{\Delta\theta}(t)g_{\theta\theta}(t)g_{\theta\Delta}(t), \\
g'_{\Delta\Delta}(t) &= rg_{\Delta\Delta}(t) - r^2\left(\sigma_Q^i(t)\right)^2\alpha_\Delta^2(t) \\
&\quad + 2a_\Delta(t)g_{\Delta\Delta}(t) + \sigma_\theta^2 g_{\theta\Delta}^2(t) + \sigma_\Delta^2(t)g_{\Delta\Delta}^2(t) + 2\sigma_\theta\sigma_{\Delta\theta}(t)g_{\theta\Delta}(t)g_{\Delta\Delta}(t), \\
g'_{\theta\Delta}(t) &= rg_{\theta\Delta}(t) - r^2\left(\sigma_Q^i(t)\right)^2\alpha_\theta(t)\alpha_\Delta(t) \\
&\quad + ag_{\theta\Delta}(t) + a_\Delta(t)g_{\theta\Delta}(t) + \sigma_\theta^2 g_{\theta\theta}(t)g_{\theta\Delta}(t) + \sigma_\Delta^2(t)g_{\Delta\Delta}(t)g_{\theta\Delta}(t) \\
&\quad + \sigma_\theta\sigma_{\Delta\theta}(t)\left[g_{\theta\theta}(t)g_{\Delta\Delta}(t) + g_{\theta\Delta}^2(t)\right].
\end{aligned} \tag{56}$$

$$\begin{aligned}
f'(t) &= r-\rho-r\ln r+rf(t)-\frac{1}{2}r^2\left(\sigma_Q^u(t)\right)^2\beta_0^2(t)+\frac{1}{2}\left(\sigma_\theta^u\right)^2\left[f_\theta^2(t)-f_{\theta\theta}(t)\right], \\
f_\theta'(t) &= rf_\theta(t)-r^2\left(\sigma_Q^u(t)\right)^2\beta_0(t)\beta_\theta(t)+af_\theta(t)+\left(\sigma_\theta^u\right)^2 f_\theta(t)f_{\theta\theta}(t), \\
f_{\theta\theta}'(t) &= rf_{\theta\theta}(t)-r^2\left(\sigma_Q^u(t)\right)^2\beta_\theta^2(t)+2af_{\theta\theta}(t)+\left(\sigma_\theta^u\right)^2 f_{\theta\theta}^2(t). \qquad (57)
\end{aligned}$$

6.7 Proof for Equilibrium Conditions on the Boundary

Boundary Conditions for the Informed Investor. The informed investor's optimization problem at the boundaries can be written as:

$$-e^{-rW^{i-}-g(T,\theta_T,\Delta_T)} = e^{-rW^{i-}}\max_{\alpha_T}\left\{-\mathbb{E}\left[e^{-r\alpha\left(P_T^+-P_T^-\right)-g(0,\theta_T,0)}\mid\mathcal{F}^i\right]\right\} \qquad (58)$$

where $x_T\sim\mathcal{N}\left(\hat{x}_T,\hat{q}_T\right)$. Solving the exponent part within the expectation operator yields:

$$-r\alpha\left(P_T^+-P_T^-\right)-g\left(0,\theta_T,0\right) = -\Phi_0-\Phi_1 x_T,$$

where $\Phi_0 = r\alpha\left\{\left[\phi(0)-\phi(T)\right]+\left[\phi_\theta(0)-\phi_\theta(T)\right]\theta_T-\phi_y\hat{x}_T+\phi_\Delta(T)\Delta_T\right\}+g(0)+g_\theta(0)\theta_T+\frac{1}{2}g_{\theta\theta}(0)\theta_T^2$, $\Phi_1=r\alpha\phi_y$. Then

$$\mathbb{E}\left[e^{-r\alpha\left(P_T^+-P_T^-\right)-g(0,\theta_T,0)}\mid\mathcal{F}^i\right] = e^{-\Phi_0-\left(\Phi_1\hat{x}_T-\frac{1}{2}\Phi_1^2\hat{q}_T\right)}=e^{Term^i},$$

where

$$\begin{aligned}
Term^i &= -r\alpha\left\{\left[\phi(0)-\phi(T)\right]+\left[\phi_\theta(0)-\phi_\theta(T)\right]\theta_T+\phi_\Delta(T)\Delta_T\right\} \\
&\quad -g(0)-g_\theta(0)\theta_T-\frac{1}{2}g_{\theta\theta}(0)\theta_T^2+\frac{1}{2}r^2\alpha^2\phi_y^2\hat{q}_T.
\end{aligned}$$

Optimization implies

$$\alpha=\frac{c_0+c_\theta\theta_T+c_\Delta\Delta_T}{r\alpha_q}$$

where $c_0=\phi(0)-\phi(T);\ c_\theta=\phi_\theta(0)-\phi_\theta(T);\ c_\Delta=\phi_\Delta(T);\ \alpha_q=\phi_y^2\hat{q}_T$. Therefore, $g(T,\theta_T,\Delta_T)=-Term^i$ gives

$$\begin{aligned}
& g(T)+g_\theta(T)\theta_T+\frac{1}{2}g_{\theta\theta}(T)\theta_T^2+g_\Delta(T)\Delta_T+\frac{1}{2}g_{\Delta\Delta}(T)\Delta_T^2+g_{\theta\Delta}(T)\theta_T\Delta_T \\
= & \frac{\left(c_0+c_\theta\theta_T+c_\Delta\Delta_T\right)^2}{2\alpha_q}+\frac{1}{2}g_{\theta\theta}(0)\theta_T^2+g_\theta(0)\theta_T+g(0).
\end{aligned}$$

Matching the coefficients yields the boundary conditions summarized in the following Lemma:

Lemma 4. *At the pre-determined announcement T (equivalent to periodical announcements nT, $n=1,2,\ldots$), the boundary conditions for informed investor's value function could be characterized* *by:*

$$
\begin{aligned}
g(T)-g(0) &= \frac{[\phi(T)-\phi(0)]^2}{2\hat{q}_T\phi_y^2},\ g_{\theta\theta}(T)-g_{\theta\theta}(0)=\frac{[\phi_\theta(T)-\phi_\theta(0)]^2}{\hat{q}_T\phi_y^2},\\
g_\theta(T)-g_\theta(0) &= \frac{[\phi(T)-\phi(0)][\phi_\theta(T)-\phi_\theta(0)]}{\hat{q}_T\phi_y^2},\ g_{\Delta\Delta}(T)=\frac{\phi_\Delta^2(T)}{\hat{q}_T\phi_y^2},\\
g_\Delta(T) &= -\frac{[\phi(T)-\phi(0)]\phi_\Delta(T)}{\hat{q}_T\phi_y^2},\ g_{\theta\Delta}(T)=-\frac{\phi_\Delta(T)[\phi_\theta(T)-\phi_\theta(0)]}{\hat{q}_T\phi_y^2}. \qquad (59)
\end{aligned}
$$

Boundary Conditions for the Uninformed Investor. Solving the exponent part within the expectation operator in Equation (58) gives:

$$
\begin{aligned}
&-r\beta\left\{[\phi(0)-\phi(T)]+\left[\phi_\theta(0)\theta_T-\phi_\theta(T)\tilde{\theta}_T\right]+\phi_y(x_T-\tilde{x}_T)\right\}-f(0)-f_\theta(0)\theta_T-\frac{1}{2}f_{\theta\theta}(0)\theta_T^2\\
=&\ -\left(\Psi_0+\Psi_2\theta_T+\Psi_{22}\theta_T^2+\Psi_1x_T\right),
\end{aligned}
$$

where $\begin{pmatrix} x_T \\ \theta_T \end{pmatrix}\sim\mathcal{N}\left(\begin{pmatrix}\tilde{x}_T\\ \tilde{\theta}_T\end{pmatrix},\begin{pmatrix}\hat{q}_T+\tilde{q}_{22,T} & -\frac{\phi_x(T)}{\phi_\theta(T)}\tilde{q}_{22,T}\\ & \frac{\phi_x^2(T)}{\phi_\theta^2(T)}\tilde{q}_{22,T}\end{pmatrix}\right)$, $\Psi_0=r\beta\left[\phi(0)-\phi(T)-\phi_\theta(T)\tilde{\theta}_T-\phi_y\tilde{x}_T\right]+f(0)$, $\Psi_2=r\beta\phi_\theta(0)+f_\theta(0)$, $\Psi_{22}=\frac{1}{2}f_{\theta\theta}(0)=\frac{1-\phi_\theta^2(T)}{2\phi_x^2(T)\tilde{q}_{22,T}}$, $\Psi_1=r\beta\phi_y$. Given $\phi_\theta(t)<0$, this implies

$$
\mathbb{E}\left[e^{-r\beta\left(P_T^+-P_T^-\right)-f(0,\theta_T)}\mid\mathcal{F}^u\right]=-\phi_{\theta,T}e^{-\Psi_0-\frac{1}{2}\bar{\Psi}}=e^{Term^u}
$$

where

$$
\begin{aligned}
\bar{\Psi} &= 2\tilde{x}_T\Psi_1-\frac{2\tilde{\theta}_T\phi_{\theta,T}\left[\Psi_1\left(\phi_{\theta,T}^2-1\right)-\Psi_2\phi_{x,T}\phi_{\theta,T}\right]}{\phi_{x,T}}-\Psi_1^2\left(\hat{q}_T+\tilde{q}_{22,T}\phi_{\theta,T}^2\right)\\
&\quad-\frac{\tilde{\theta}_T^2\left(\phi_{\theta,T}^2-1\right)\phi_{\theta,T}^2}{\tilde{q}_{22,T}\phi_{x,T}^2}+2\tilde{q}_{22,T}\Psi_1\Psi_2\phi_{x,T}\phi_{\theta,T}-\tilde{q}_{22,T}\Psi_2^2\phi_{x,T}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
Term^u &= -r\beta\left[\ \phi(0)-\phi(T)+f_\theta(0)\tilde{q}_{22,T}\phi_{x,T}\left(\phi_{\theta,T}\phi_y-\phi_{x,T}\phi_{\theta,0}\right)+\tilde{\theta}_T\phi_{\theta,T}\left(\phi_{\theta,T}\phi_{\theta,0}+\frac{1-\phi_{\theta,T}^2}{\phi_{x,T}}\phi_y-1\right)\ \right]\\
&\quad+\frac{1}{2}r^2\beta^2\left[\ \phi_y^2\left(\hat{q}_T+\tilde{q}_{22,T}\phi_{\theta,T}^2\right)+\tilde{q}_{22,T}\phi_{x,T}^2\phi_{\theta,0}^2-2\tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\phi_{\theta,0}\phi_y\ \right]\\
&\quad+\frac{\left(f_\theta(0)\tilde{q}_{22,T}\phi_{x,T}^2-\tilde{\theta}_T\phi_{\theta,T}^2\right)^2-\tilde{\theta}_T^2\phi_{\theta,T}^2}{2\tilde{q}_{22,T}\phi_{x,T}^2}-f(0)+\ln\left(-\phi_{\theta,T}\right).
\end{aligned}
$$

Using Equation (14), solving the optimization problem gives

$$
\beta = \frac{d_0+d_\theta\tilde{\theta}_T}{r\beta_q}=\frac{d_0+d_\theta\left(\theta_T+\frac{\phi_{x,T}}{\phi_{\theta,T}}\Delta_T\right)}{r\beta_q}=\frac{d_0+d_\theta\theta_T+d_\theta\frac{\phi_{x,T}}{\phi_{\theta,T}}\Delta_T}{r\beta_q}
$$

where

$$
\begin{aligned}
d_0 &= \phi(0) - \phi(T) + f_\theta(0)\,\tilde{q}_{22,T}\phi_{x,T}\left(\phi_{\theta,T}\phi_y - \phi_{x,T}\phi_{\theta,0}\right) \\
d_\theta &= \phi_{\theta,T}\left(\phi_{\theta,T}\phi_{\theta,0} + \frac{1-\phi_{\theta,T}^2}{\phi_{x,T}}\phi_y - 1\right) \\
\beta_q &= \phi_y^2\hat{q}_T + \left(\phi_{x,T}\phi_{\theta,0} - \phi_y\phi_{\theta,T}\right)^2\tilde{q}_{22,T}.
\end{aligned}
$$

Therefore, $f\left(T, \tilde{\theta}_T\right) = -Term^u$ gives

$$
\begin{aligned}
& f(T) + f_\theta(T)\,\tilde{\theta}_T + \frac{1}{2}f_{\theta\theta}(T)\,\tilde{\theta}_T^2 \\
= \; & \frac{\left(d_0 + d_\theta\tilde{\theta}_T\right)^2}{2\beta_q} - \frac{\left(f_\theta(0)\,\tilde{q}_{22,T}\phi_{x,T}^2 - \tilde{\theta}_T\phi_{\theta,T}^2\right)^2 - \tilde{\theta}_T^2\phi_{\theta,T}^2}{2\tilde{q}_{22,T}\phi_{x,T}^2} + f(0) - \ln\left(-\phi_{\theta,T}\right).
\end{aligned}
$$

Matching coefficients and substituting d_0, d_θ, β_q back yields the boundary conditions summarized in the following lemma:

Lemma 5. *At the pre-determined announcement T (equivalent to periodical announcements nT, $n = 1, 2, \ldots$), the boundary conditions for uninformed investor's value function could be characterized by:*

$$
\begin{aligned}
f(T) - f(0) &= \frac{1}{2\beta_q}\left[\tilde{q}_{22,T}f_\theta(0)\,\phi_{x,T}\left(\phi_{\theta,T}\phi_y - \phi_{\theta,0}\phi_{x,T}\right) - \phi_T + \phi_0\right]^2 \\
&\quad -\frac{1}{2}f_\theta^2(0)\,\tilde{q}_{22,T}\phi_{x,T}^2 - ln\left|\phi_{\theta,T}\right|, && (60) \\
f_{\theta\theta}(T) - f_{\theta\theta}(0) &= \frac{1}{\tilde{q}_{22,T}\phi_{x,T}^2\beta_q}\left\{\begin{array}{c} \tilde{q}_{22,T}\phi_{x,T}\left(\phi_{\theta,0} - \phi_{\theta,T}\right)\Big[\phi_{x,T}\left(\phi_{\theta,0}\left(2\phi_{\theta,T}^2 - 1\right) - \phi_{\theta,T}\right) \\ -2\phi_{\theta,T}\left(\phi_{\theta,T}^2 - 1\right)\phi_y\Big] - \hat{q}_T\left(\phi_{\theta,T}^2 - 1\right)^2\phi_y^2 \end{array}\right\} && (61) \\
f_\theta(T) - f_\theta(0) &= \frac{1}{\phi_{x,T}\beta_q}\left\{\begin{array}{c} \hat{q}_T f_\theta(0)\left(\phi_{\theta,T}^2 - 1\right)\phi_y^2\phi_{x,T} \\ -\tilde{q}_{22,T}f_\theta(0)\,\phi_{x,T}^2\left(\phi_{\theta,0} - \phi_{\theta,T}\right)\left(\phi_{\theta,0}\phi_{x,T} - \phi_{\theta,T}\phi_y\right) \\ +\phi_{\theta,T}\left(\phi_0 - \phi_T\right)\left[\phi_{x,T}\left(\phi_{\theta,0}\phi_{\theta,T} - 1\right) - \left(\phi_{\theta,T}^2 - 1\right)\phi_y\right] \end{array}\right\}. && (62)
\end{aligned}
$$

6.8 Solving for Time-Varying Price Sensitivities

Denote $l_0 = \phi(T) - \phi(0)$; $l_x = \phi_x(T) - \phi_x(0)$; $l_\Delta = \phi_\Delta(T) - \phi_\Delta(0)$; $l_\theta = \phi_\theta(T) - \phi_\theta(0)$; and $l_\Delta = -l_x$. Rewrite the coefficients in Lemma 3 as follows:

$$
\begin{aligned}
c_0 &= -l_0;\; c_\theta = -l_\theta;\; c_\Delta = \phi_{\Delta,T};\; \alpha_q = \phi_y^2\hat{q}_T. \\
g(T) &= g(0) + \frac{l_0^2}{2\alpha_q};\; g_{\theta\theta}(T) = g_{\theta\theta}(0) + \frac{l_\theta^2}{\alpha_q};\; g_\theta(T) = g_\theta(0) + \frac{l_0 l_\theta}{\alpha_q}; \\
g_{\Delta\Delta}(T) &= \frac{\phi_{\Delta,T}^2}{\alpha_q};\; g_\Delta(T) = -\frac{l_0\phi_{\Delta,T}}{\alpha_q};\; g_{\theta\Delta}(T) = -\frac{l_\theta\phi_{\Delta,T}}{\alpha_q};
\end{aligned}
$$

$$
\begin{aligned}
d_0 &= \tilde{q}_{22,T} f_\theta(0)\,\phi_{x,T}\left(\phi_{\theta,T}\phi_{\Delta,T}+l_\theta\phi_{x,T}\right)-l_0;\\
d_\theta &= -\phi_{\theta,T}^2 l_\theta-\frac{\phi_{\theta,T}\phi_{\Delta,T}\left(\phi_{\theta,T}^2-1\right)}{\phi_{x,T}};\\
\beta_q &= \tilde{q}_{22,T}\left(\phi_{\Delta,T}\phi_{\theta,T}+l_\theta\phi_{x,T}\right)^2+\hat{q}_T\phi_y^2;\\
f(T) &= f(0)-\ln\left(-\phi_{\theta,T}\right)-\frac{1}{2}f_\theta^2(0)\,\phi_{x,T}^2\tilde{q}_{22,T}+\frac{\left[l_0-\tilde{q}_{22,T}f_\theta(0)\,\phi_{x,T}\left(\phi_{\Delta,T}\phi_{\theta,T}+l_\theta\phi_{x,T}\right)\right]^2}{2\beta_q};\\
f_{\theta\theta}(T) &= f_{\theta\theta}(0)+\frac{\tilde{q}_{22,T}l_\theta\phi_{x,T}\left[2\phi_{\Delta,T}\phi_{\theta,T}\left(\phi_{\theta,T}^2-1\right)+\left(2\phi_{\theta,T}^2-1\right)l_\theta\phi_{x,T}\right]-\hat{q}_T\left(\phi_{\theta,T}^2-1\right)^2\phi_y^2}{\tilde{q}_{22,T}\phi_{x,T}^2\beta_q};\\
f_\theta(T) &= f_\theta(0)+\frac{\left\{\begin{array}{c}-\tilde{q}_{22,T}f_\theta(0)\,l_\theta\phi_{x,T}^2\left(\phi_{\Delta,T}\phi_{\theta,T}+l_\theta\phi_{x,T}\right)+\hat{q}_T f_\theta(0)\left(\phi_{\theta,T}^2-1\right)\phi_y^2\phi_{x,T}\\ +l_0\phi_{\theta,T}\left[\phi_{\Delta,T}\left(\phi_{\theta,T}^2-1\right)+\phi_{\theta,T}l_\theta\phi_{x,T}\right]\end{array}\right\}}{\phi_{x,T}\beta_q}.
\end{aligned}
$$

Matching coefficients in Equation (36) would immediately give the following equations:

$$
\begin{aligned}
\frac{(1-\omega)\,c_0}{\alpha_q}+\frac{\omega d_0}{\beta_q} &= 0,\\
\frac{(1-\omega)\,c_\Delta}{\alpha_q}+\frac{\omega d_\theta\frac{\phi_{x,T}}{\phi_{\theta,T}}}{\beta_q} &= 0,\\
\frac{(1-\omega)\,c_\theta}{\alpha_q}+\frac{\omega d_\theta}{\beta_q} &= r.
\end{aligned}
$$

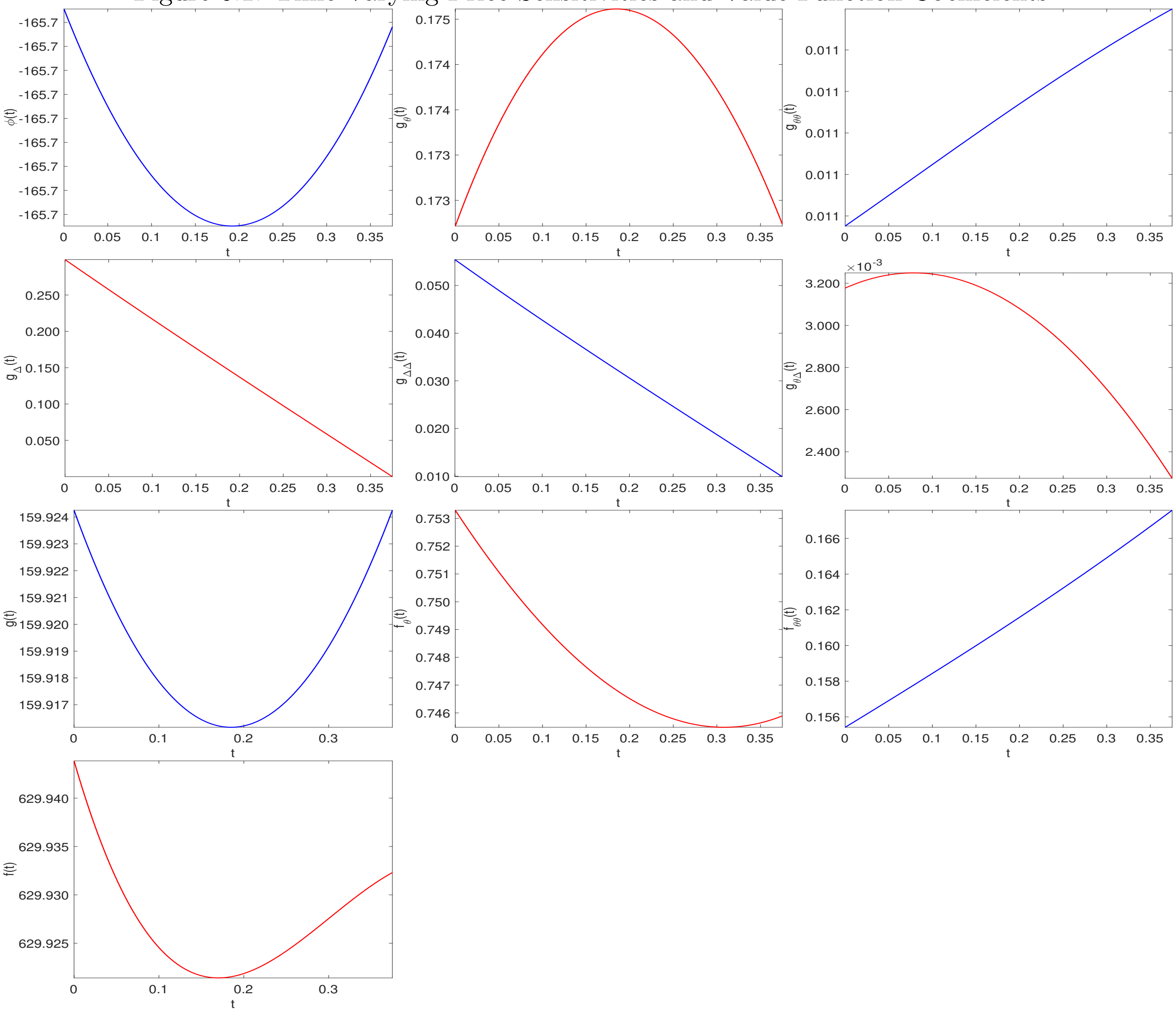
Combining with the above conditions would yield the results in the main text. Note that this solution is a special case of the general setting of Appendix 6.9.

Lemma 6. *At the pre-determined announcement T (equivalent to periodical announcements nT, $n=1,2,\ldots$), the equilibrium pricing function sensitivities (coefficients of state variables) satisfy*

$$
\begin{aligned}
\phi_\theta(0) &= -\left[\frac{r\phi_{\Delta,T}\phi_y\tilde{q}_{22,T}}{\omega}+\frac{(1-\omega)\phi_{\Delta,T}}{r\omega\phi_y\left(\phi_y-\phi_{\Delta,T}\right)^2\hat{q}_T}+\frac{r\phi_y^2\hat{q}_T}{\omega-1}\right], && (63)\\
\phi_\theta(T) &= -\frac{\phi_{\Delta,T}\left[1-\omega+r^2\phi_y^2\left(\phi_y-\phi_{\Delta,T}\right)^2\hat{q}_T\tilde{q}_{22,T}\right]}{r\omega\phi_y^2\left(\phi_y-\phi_{\Delta,T}\right)\hat{q}_T}, && (64)\\
\phi(T)-\phi(0) &= -\frac{f_\theta(0)\,r\omega\phi_y^2\left(\phi_y-\phi_{\Delta,T}\right)^2\hat{q}_T\tilde{q}_{22,T}}{r^2\phi_y^2\left(\phi_y-\phi_{\Delta,T}\right)^2\hat{q}_T\tilde{q}_{22,T}-\omega+1}. && (65)
\end{aligned}
$$

Simulated Price Sensitivities and Value Function Coefficients. This figure displays the calibrated time-varying coefficients of value function and price sensitivities.

Figure 6.1: Time Varying Price Sensitivities and Value Function Coefficients



Online Appendix

6.9 Proof for Value Function Assumptions

This Appendix proves the sufficient conditions for letting the value functions do not depend on the state variable $\hat{x}_t$ and $\tilde{x}_t$.

Assumption 1. *In order to ensure the value functions do not depend on $\hat{x}_t$ or $\tilde{x}_t$, we impose the following assumption: Given ϕ_y is a constant, assume $\phi_X(0) \neq 0$, $\phi_x(T) \neq 0$ (or $\phi_\theta(T) \neq \pm 1$, $\tilde{q}_{22,T} \neq 0$), and the following sufficient condition:*

$$f_{\theta\theta}(0) = \frac{1 - \phi_\theta^2(T)}{\phi_x^2(T)\,\tilde{q}_{22,T}}. \tag{66}$$

Proof. First start from the generalized full model without assuming those conditions, and prove the sufficient conditions in equilibrium on the boundary.

Optimization Problem of Informed Investor on the Boundary.

Now consider an informed investor's excess return follows the generalized law of motion:

$$dQ_t^i = [e_0(t) + e_\theta(t)\,\theta_t + e_\Delta(t)\,\Delta_t + e_x(t)\,\hat{x}_t]\,dt + b_D^i(t)\,d\hat{B}_{D,t} + b_s^i(t)\,d\hat{B}_{s,t} + b_\theta^i(t)\,dB_{\theta,t} \tag{67}$$

where the additional term

$$e_x(t) = \phi_D + \phi_y'(t) - (b+r)\,\phi_y(t). \tag{68}$$

Let the informed investor's value function be of the generalized form

$$J\left(t, W^i, \theta, \Delta, \hat{x}\right) = -e^{-rW^i - g(t,\theta,\Delta,\hat{x})}$$

where

$$\begin{aligned} g(t,\theta,\Delta,\hat{x}) &= g(t) + g_\theta(t)\,\theta_t + \frac{1}{2}g_{\theta\theta}(t)\,\theta_t^2 + g_x(t)\,\hat{x}_t + \frac{1}{2}g_{xx}(t)\,\hat{x}_t^2 + g_{\theta x}(t)\,\theta_t\hat{x}_t \\ &\quad + g_\Delta(t)\,\Delta_t + \frac{1}{2}g_{\Delta\Delta}(t)\,\Delta_t^2 + g_{\theta\Delta}(t)\,\theta_t\Delta_t + g_{x\Delta}(t)\,\hat{x}_t\Delta_t. \end{aligned}$$

The informed investor's optimization problem at the boundaries is

$$\begin{aligned} -e^{-rW^i - g(T,\theta_T,\Delta_T,\hat{x}_T)} &= \max_\alpha \left\{ -\mathbb{E}\left[e^{-rW^{i\prime} - g(0,\theta_T,0,x_T)} \mid \mathcal{F}^i \right] \right\} \\ &= e^{-rW^i} \max_\alpha \left\{ -\mathbb{E}\left[e^{-r\alpha\left(P_T^+ - P_T^-\right) - g(0,\theta_T,0,x_T)} \mid \mathcal{F}^i \right] \right\} \end{aligned}$$

where $x_T \sim \mathcal{N}(\hat{x}_T, \hat{q}_T)$.

Taking out the exponent part,

$$
\begin{aligned}
& -r\alpha\left\{\left[\phi(0)-\phi(T)\right]+\left[\phi_\theta(0)-\phi_\theta(T)\right]\theta_T+\phi_X(0)\,x_T-\phi_X(T)\,\hat{x}_T+\phi_\Delta(T)\,\Delta_T\right\} \\
& -g(0)-g_\theta(0)\,\theta_T-\frac{1}{2}g_{\theta\theta}(0)\,\theta_T^2-g_x(0)\,x_T-\frac{1}{2}g_{xx}(0)\,x_T^2-g_{\theta x}(0)\,\theta_T x_T \\
= \; & -\Phi_0-\Phi_1 x_T-\frac{1}{2}g_{xx}(0)\,x_T^2,
\end{aligned}
$$

where $\Phi_0=r\alpha\left\{\left[\phi(0)-\phi(T)\right]+\left[\phi_\theta(0)-\phi_\theta(T)\right]\theta_T-\phi_X(T)\,\hat{x}_T+\phi_\Delta(T)\,\Delta_T\right\}+g(0)+g_\theta(0)\,\theta_T+\frac{1}{2}g_{\theta\theta}(0)\,\theta_T^2$, $\Phi_1=r\alpha\phi_X(0)+g_x(0)+g_{\theta x}(0)\,\theta_T$. It is easy to see that only the second row is unknown. Denote $\hat{m}=\frac{1}{1+\hat{q}_T g_{xx}(0)}$, then

$$
\mathbb{E}\left[e^{-r\alpha\left(P_T^+-P_T^-\right)-g\left(0,\theta_T,0,x_T\right)}\mid\mathcal{F}^i\right]=e^{-\Phi_0}\times\sqrt{\hat{m}}e^{-\frac{1}{2}\hat{m}\bar{\Phi}},
$$

where

$$
\begin{aligned}
\bar{\Phi} &= 2\Phi_1\hat{x}_T-\hat{q}_T\Phi_1^2+g_{xx}(0)\,\hat{x}_T^2 \\
&= 2\left[r\alpha\phi_X(0)+g_x(0)+g_{\theta x}(0)\,\theta_T\right]\hat{x}_T+g_{xx}(0)\,\hat{x}_T^2 \\
&\quad -\hat{q}_T\left\{r^2\alpha^2\phi_X^2(0)+\left[g_x(0)+g_{\theta x}(0)\,\theta_T\right]^2+2r\alpha\phi_X(0)\left[g_x(0)+g_{\theta x}(0)\,\theta_T\right]\right\}.
\end{aligned}
$$

Therefore, $\mathbb{E}\left[e^{-r\alpha\left(P_T^+-P_T^-\right)-g\left(0,\theta_T,0,x_T\right)}\mid\mathcal{F}^i\right]=e^{Term^i}$, where

$$
\begin{aligned}
Term^i &= -r\alpha\left\{\begin{array}{c}\left[\phi(0)-\phi(T)-\hat{m}\hat{q}_T\phi_X(0)\,g_x(0)\right]+\left[\phi_\theta(0)-\phi_\theta(T)-\hat{m}\hat{q}_T\phi_X(0)\,g_{\theta x}(0)\right]\theta_T \\ +\left[\hat{m}\phi_X(0)-\phi_X(T)\right]\hat{x}_T+\phi_\Delta(T)\,\Delta_T\end{array}\right\} \\
&\quad +\frac{1}{2}\alpha^2\hat{m}\hat{q}_T r^2\phi_X^2(0)+\frac{1}{2}\left[\hat{m}\hat{q}_T g_{\theta x}^2(0)-g_{\theta\theta}(0)\right]\theta_T^2+\left[\hat{m}\hat{q}_T g_x(0)\,g_{\theta x}(0)-g_\theta(0)\right]\theta_T \\
&\quad -\frac{1}{2}\hat{m}g_{xx}(0)\,\hat{x}_T^2-\hat{m}\left[g_x(0)+g_{\theta x}(0)\,\theta_T\right]\hat{x}_T+\frac{1}{2}\hat{m}\hat{q}_T g_x^2(0)+\frac{1}{2}\ln\hat{m}-g(0).
\end{aligned}
$$

Optimization implies: $\alpha=\frac{c_0+c_\theta\theta_T+c_x\hat{x}_T+c_\Delta\Delta_T}{r\alpha_q}$, where

$$
\begin{aligned}
c_0 &= \phi(0)-\phi(T)-\hat{m}\hat{q}_T\phi_X(0)\,g_x(0) \\
c_\theta &= \phi_\theta(0)-\phi_\theta(T)-\hat{m}\hat{q}_T\phi_X(0)\,g_{\theta x}(0) \\
c_x &= \hat{m}\phi_X(0)-\phi_X(T) \\
c_\Delta &= \phi_\Delta(T) \\
\alpha_q &= \hat{m}\hat{q}_T\phi_X^2(0).
\end{aligned}
$$

Note that unless $c_x=0$, $g(T,\theta_T,\Delta_T,\hat{x}_T)$ will in general be a quadratic function of $\theta_T,\Delta_T,\hat{x}_T$, because $Term^i$ contains α^2. This means that it is not obvious simply assume away the dependence of $g(t,\theta,\Delta,\hat{x})$ on $\hat{x}$. Therefore, $g(T,\theta_T,\Delta_T,\hat{x}_T) = -Term^i$ gives

$$\begin{aligned}
& g(T) + g_\theta(T)\theta_T + \frac{1}{2}g_{\theta\theta}(T)\theta_T^2 + g_x(T)\hat{x}_T + \frac{1}{2}g_{xx}(T)\hat{x}_T^2 + g_{\theta x}(T)\theta_T\hat{x}_T \\
& + g_\Delta(T)\Delta_T + \frac{1}{2}g_{\Delta\Delta}(T)\Delta_T^2 + g_{\theta\Delta}(T)\theta_T\Delta_T + g_{x\Delta}(T)\hat{x}_t\Delta_T \\
= & \frac{(c_0 + c_\theta\theta_T + c_x\hat{x}_T + c_\Delta\Delta_T)^2}{2\alpha_q} - \frac{1}{2}\left[\hat{m}\hat{q}_T g_{\theta x}^2(0) - g_{\theta\theta}(0)\right]\theta_T^2 - \left[\hat{m}\hat{q}_T g_x(0) g_{\theta x}(0) - g_\theta(0)\right]\theta_T \\
& + \frac{1}{2}\hat{m}g_{xx}(0)\hat{x}_T^2 + \hat{m}\left[g_x(0) + g_{\theta x}(0)\theta_T\right]\hat{x}_T - \frac{1}{2}\hat{m}\hat{q}_T g_x^2(0) - \frac{1}{2}\ln\hat{m} + g(0).
\end{aligned}$$

Matching the coefficients gives the following boundary conditions:

$$\begin{aligned}
g(T) - g(0) &= \frac{[\phi(T) - \phi(0)]^2}{2\hat{m}\hat{q}_T\phi_X^2(0)} - \frac{g_x(0)[\phi(T) - \phi(0)]}{\phi_X(0)} - \frac{1}{2}\ln(\hat{m}), \\
g_{\theta\theta}(T) - g_{\theta\theta}(0) &= \frac{[\phi(T) - \phi(0) - \hat{m}\hat{q}_T g_x(0)\phi_X(0)][\phi_\theta(T) - \phi_\theta(0) - \hat{m}\hat{q}_T g_{\theta x}(0)\phi_X(0)]}{\hat{m}\hat{q}_T\phi_X^2(0)} \\
& \quad -\hat{m}\hat{q}_T g_{\theta x}^2(0), \\
g_\theta(T) - g_\theta(0) &= \frac{[\phi(T) - \phi(0) - \hat{m}\hat{q}_T g_x(0)\phi_X(0)][\phi_\theta(T) - \phi_\theta(0) - \hat{m}\hat{q}_T g_{\theta x}(0)\phi_X(0)]}{\hat{m}\hat{q}_T\phi_X^2(0)} \\
& \quad -\hat{m}\hat{q}_T g_x(0) g_{\theta x}(0), \\
g_{xx}(T) - g_{xx}(0) &= \frac{\phi_X^2(T) + \hat{m}\phi_X(0)[\phi_X(0) - 2\phi_X(T)]}{\hat{m}\hat{q}_T\phi_X^2(0)}, \\
g_x(T) - g_x(0) &= \frac{[\phi(T) - \phi(0)]\phi_X(T) - \hat{m}\phi_X(0)[\phi(T) - \phi(0) - \hat{q}_T g_x(0)\phi_X(T)]}{\hat{m}\hat{q}_T\phi_X^2(0)}, \\
g_{\theta x}(T) - g_{\theta x}(0) &= (\hat{m} - 1) g_{\theta x}(0) + \frac{[\phi_\theta(T) - \phi_\theta(0) + \hat{m}g_{\theta x}(0)\phi_X(0)\hat{q}_T][\phi_X(T) - \hat{m}\phi_X(0)]}{\hat{m}\hat{q}_T\phi_X^2(0)}, \\
g_{\Delta\Delta}(T) &= \frac{\phi_\Delta^2(T)}{\hat{m}\hat{q}_T\phi_X^2(0)}, \\
g_\Delta(T) &= -\frac{[\phi(T) - \phi(0) + \hat{m}\hat{q}_T g_x(0)\phi_X(0)]\phi_\Delta(T)}{\hat{m}\hat{q}_T\phi_X^2(0)}, \\
g_{x\Delta}(T) &= -\frac{[\phi_X(T) - \hat{m}\phi_X(0)]\phi_\Delta(T)}{\hat{m}\hat{q}_T\phi_X^2(0)}, \\
g_{\theta\Delta}(T) &= -\frac{\phi_\Delta(T)[\phi_\theta(T) - \phi_\theta(0) + \hat{m}g_{\theta x}(0)\phi_X(0)\hat{q}_T]}{\hat{m}\hat{q}_T\phi_X^2(0)}.
\end{aligned}$$

Optimization Problem of Uninformed Investor on the Boundary.

Guess the uninformed investor's value function as a general form: $V\left(t, W^u, \tilde{\theta}, \tilde{x}\right) = -e^{-rW^u - f\left(t,\tilde{\theta},\tilde{x}\right)}$, where $f\left(t,\tilde{\theta},\tilde{x}\right) = f(t) + f_\theta(t)\tilde{\theta}_t + \frac{1}{2}f_{\theta\theta}(t)\tilde{\theta}_t^2 + f_x(t)\tilde{x}_t + \frac{1}{2}f_{xx}(t)\tilde{x}_t^2 + f_{\theta x}(t)\tilde{\theta}_t\tilde{x}_t$.

Therefore, the maximization problem is written as

$$-e^{-rW^u - f\left(T,\tilde{\theta}_T,\tilde{x}_T\right)} = e^{-rW^u}\max_\beta \mathbb{E}\left[-e^{-r\beta\left(P_T^+ - P_T^-\right) - f(0,\theta_T,x_T)} \mid \mathcal{F}^u\right]$$

where $\begin{pmatrix} x_T \\ \theta_T \end{pmatrix} \sim \mathcal{N}\left(\left(\begin{pmatrix} \tilde{x}_T \\ \tilde{\theta}_T \end{pmatrix}, \begin{pmatrix} \hat{q}_T + \tilde{q}_{22,T} & -\frac{\phi_x(T)}{\phi_\theta(T)}\tilde{q}_{22,T} \\ & \frac{\phi_x^2(T)}{\phi_\theta^2(T)}\tilde{q}_{22,T} \end{pmatrix}\right)\right)$. Focus on the exponent part:

$$\begin{aligned}
& -r\beta\left\{[\phi(0)-\phi(T)] + \left[\phi_\theta(0)\theta_T - \phi_\theta(T)\tilde{\theta}_T\right] + [\phi_X(0)x_T - \phi_X(T)\tilde{x}_T]\right\} \\
& -f(0) - f_\theta(0)\theta_T - \frac{1}{2}f_{\theta\theta}(0)\theta_T^2 - f_x(0)x_T - \frac{1}{2}f_{xx}(0)x_T^2 - f_{\theta x}(0)\theta_T x_T \\
= & -\left(\Psi_0 + \Psi_2\theta_T + \Psi_{22}\theta_T^2 + \Psi_1 x_T + \Psi_{11}x_T^2 + \Psi_{12}\theta_T x_T\right),
\end{aligned}$$

where $\Psi_0 = r\beta\left[\phi(0) - \phi(T) - \phi_\theta(T)\tilde{\theta}_T - \phi_X(T)\tilde{x}_T\right] + f(0)$, $\Psi_2 = r\beta\phi_\theta(0) + f_\theta(0)$, $\Psi_{22} = \frac{1}{2}f_{\theta\theta}(0)$, $\Psi_1 = r\beta\phi_X(0) + f_x(0)$, $\Psi_{11} = \frac{1}{2}f_{xx}(0)$, $\Psi_{12} = f_{\theta x}(0)$.

Denote $\tilde{m} = \frac{1}{\tilde{q}_{22,T}\phi_{x,T}^2\left[2\Psi_{22}(2\hat{q}_T\Psi_{11}+1) - \hat{q}_T\Psi_{12}^2\right] + \phi_{\theta,T}^2\left[2\Psi_{11}(\hat{q}_T+\tilde{q}_{22,T})+1\right] - 2\tilde{q}_{22,T}\Psi_{12}\phi_{\theta,T}\phi_{x,T}}$, this simplifies:

$$\mathbb{E}\left[e^{-r\beta\left(P_T^+ - P_T^-\right) - f(0,\theta_T,x_T)} \mid \mathcal{F}^u\right] = -e^{-\Psi_0}\left|\phi_{\theta,T}\right|\sqrt{\tilde{m}}e^{-\frac{1}{2}\tilde{m}\bar{\Psi}}$$

where

$$\begin{aligned}
& \bar{\Psi} \\
= & \tilde{x}_T^2\left[2\Psi_{11}\phi_{\theta,T}^2 - \tilde{q}_{22,T}\left(\Psi_{12}^2 - 4\Psi_{11}\Psi_{22}\right)\phi_{x,T}^2\right] + \tilde{\theta}_T^2\phi_{\theta,T}^2\left[2\Psi_{22}\left(2\Psi_{11}(\hat{q}_T+\tilde{q}_{22,T})+1\right) - \Psi_{12}^2(\hat{q}_T+\tilde{q}_{22,T})\right] \\
& +2\tilde{x}_T\tilde{\theta}_T\phi_{\theta,T}\left[\Psi_{12}\phi_{\theta,T} - \tilde{q}_{22,T}\left(\Psi_{12}^2 - 4\Psi_{11}\Psi_{22}\right)\phi_{x,T}\right] \\
& +\tilde{x}_T\left[-2\tilde{q}_{22,T}\left(\Psi_2\Psi_{12} - 2\Psi_1\Psi_{22}\right)\phi_{x,T}^2 + 2\tilde{q}_{22,T}\left(2\Psi_2\Psi_{11} - \Psi_1\Psi_{12}\right)\phi_{\theta,T}\phi_1 + 2\Psi_1\phi_{\theta,T}^2\right] \\
& +2\tilde{\theta}_T\phi_{\theta,T}\left[\phi_{\theta,T}\left(\Psi_2\left(2\Psi_{11}(\hat{q}_T+\tilde{q}_{22,T})+1\right) - \Psi_1\Psi_{12}(\hat{q}_T+\tilde{q}_{22,T})\right) + \tilde{q}_{22,T}\left(2\Psi_1\Psi_{22} - \Psi_2\Psi_{12}\right)\phi_{x,T}\right] \\
& -\Psi_1^2\phi_{\theta,T}^2(\hat{q}_T+\tilde{q}_{22,T}) - \tilde{q}_{22,T}\phi_{x,T}^2\left[2\hat{q}_T\Psi_{22}\Psi_1^2 - 2\hat{q}_T\Psi_2\Psi_{12}\Psi_1 + \Psi_2^2\left(2\hat{q}_T\Psi_{11}+1\right)\right] + 2\tilde{q}_{22,T}\Psi_1\Psi_2\phi_{x,T}\phi_{\theta,T}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[e^{-r\beta\left(P_T^+ - P_T^-\right) - f(0,\theta_T,x_T)} \mid \mathcal{F}^u\right] = e^{Term^u},$$

where

$$\begin{aligned}
& Term^u \\
= & -r\beta\left\{\begin{array}{c}
\phi(0) - \phi(T) + \tilde{m}f_{\theta,0}\tilde{q}_{22,T}\left[\phi_{x,T}^2\left(\hat{q}_T\Psi_{12}\phi_{X,0} - \phi_{\theta,0}\left(2\hat{q}_T\Psi_{11}+1\right)\right) + \phi_{x,T}\phi_{\theta,T}\phi_{X,0}\right] \\
+\tilde{m}f_{x,0}\left[\hat{q}_T\tilde{q}_{22,T}\phi_{x,T}^2\left(\Psi_{12}\phi_{\theta,0} - 2\Psi_{22}\phi_{X,0}\right) - \phi_{\theta,T}^2(\hat{q}_T+\tilde{q}_{22,T})\phi_{X,0} + \tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\phi_{\theta,0}\right] \\
+\tilde{m}\tilde{x}_T\Big[-\phi_{X,T} + \tilde{q}_{22,T}\phi_{x,T}^2\left(2\Psi_{22}\phi_{X,0} - \Psi_{12}\phi_{\theta,0}\right) + \tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\left(2\Psi_{11}\phi_{\theta,0} - \Psi_{12}\phi_{X,0}\right) \\
+\phi_{\theta,T}^2\phi_{X,0}\Big] + \tilde{m}\tilde{\theta}_T\Big[-\phi_{\theta,T} + \phi_{\theta,T}^2\left(\phi_{\theta,0}\left(2\Psi_{11}(\hat{q}_T+\tilde{q}_{22,T})+1\right) - \Psi_{12}(\hat{q}_T+\tilde{q}_{22,T})\phi_{X,0}\right) \\
+\tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\left(2\Psi_{22}\phi_{X,0} - \Psi_{12}\phi_{\theta,0}\right)\Big]
\end{array}\right\} \\
& +\frac{1}{2}\tilde{m}r^2\beta^2\left\{\begin{array}{c}
-\tilde{q}_{22,T}\phi_{x,T}^2\left[\phi_{\theta,0}\left(2\hat{q}_T\Psi_{12}\phi_{X,0} - \phi_{\theta,0}\left(2\hat{q}_T\Psi_{11}+1\right)\right) - 2\hat{q}_T\Psi_{22}\phi_{X,0}^2\right] \\
+\phi_{\theta,T}^2(\hat{q}_T+\tilde{q}_{22,T})\phi_{X,0}^2 - 2\tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\phi_{\theta,0}\phi_{X,0}
\end{array}\right\}
\end{aligned}$$

$$-\frac{1}{2}\tilde{m}\left[2\Psi_{11}\phi_{\theta,T}^{2}-\tilde{q}_{22,T}\left(\Psi_{12}^{2}-4\Psi_{11}\Psi_{22}\right)\phi_{x,T}^{2}\right]\tilde{x}_{T}^{2}-\tilde{m}\phi_{\theta,T}\left[\Psi_{12}\phi_{\theta,T}-\tilde{q}_{22,T}\left(\Psi_{12}^{2}-4\Psi_{11}\Psi_{22}\right)\phi_{x,T}\right]\tilde{x}_{T}\tilde{\theta}_{T}$$
$$-\frac{1}{2}\tilde{m}\phi_{\theta,T}^{2}\left[2\Psi_{22}\left(2\Psi_{11}(\hat{q}_{T}+\tilde{q}_{22,T})+1\right)-\Psi_{12}^{2}(\hat{q}_{T}+\tilde{q}_{22,T})\right]\tilde{\theta}_{T}^{2}$$
$$+\tilde{m}\left[f_{\theta,0}\tilde{q}_{22,T}\left(\Psi_{12}\phi_{x,T}^{2}-2\Psi_{11}\phi_{x,T}\phi_{\theta,T}\right)-f_{x,0}\left(2\tilde{q}_{22,T}\Psi_{22}\phi_{x,T}^{2}-\tilde{q}_{22,T}\Psi_{12}\phi_{x,T}\phi_{\theta,T}+\phi_{\theta,T}^{2}\right)\right]\tilde{x}_{T}$$
$$-\tilde{m}\phi_{\theta,T}\Big[f_{\theta,0}\left(\phi_{\theta,T}\left(2\Psi_{11}(\hat{q}_{T}+\tilde{q}_{22,T})+1\right)-\tilde{q}_{22,T}\Psi_{12}\phi_{x,T}\right)$$
$$+f_{x,0}\left(2\tilde{q}_{22,T}\Psi_{22}\phi_{x,T}-\Psi_{12}\phi_{\theta,T}(\hat{q}_{T}+\tilde{q}_{22,T})\right)\Big]\tilde{\theta}_{T}$$
$$-\frac{1}{2}\tilde{m}\left[-f_{x,0}^{2}\left(2\hat{q}_{T}\tilde{q}_{22,T}\Psi_{22}\phi_{x,T}^{2}+\phi_{\theta,T}^{2}(\hat{q}_{T}+\tilde{q}_{22,T})\right)+f_{\theta,0}^{2}\tilde{q}_{22,T}\phi_{x,T}^{2}\left(2\hat{q}_{T}\Psi_{11}+1\right)\right]$$
$$-\tilde{m}f_{\theta,0}f_{x,0}\tilde{q}_{22,T}\phi_{x,T}\left(\hat{q}_{T}\Psi_{12}\phi_{x,T}+\phi_{\theta,T}\right)-f(0)+\frac{1}{2}\ln\tilde{m}+\ln\left|\phi_{\theta,T}\right|.$$

Optimization implies

$$\beta = \frac{d_0+d_\theta\tilde{\theta}_T+d_x\tilde{x}_T}{r\beta_q}=\frac{d_0+d_\theta\theta_T+d_x\hat{x}_T+\left(d_\theta\frac{\phi_{x,T}}{\phi_{\theta,T}}-d_x\right)\Delta_T}{r\beta_q},$$

where

$$\begin{aligned}
d_0 &= \phi(0)-\phi(T)+\tilde{m}f_{\theta,0}\tilde{q}_{22,T}\left[\phi_{x,T}^{2}\left(\hat{q}_{T}\Psi_{12}\phi_{X,0}-\phi_{\theta,0}\left(2\hat{q}_{T}\Psi_{11}+1\right)\right)+\phi_{x,T}\phi_{\theta,T}\phi_{X,0}\right]\\
&\quad+\tilde{m}f_{x,0}\left[\hat{q}_{T}\tilde{q}_{22,T}\phi_{x,T}^{2}\left(\Psi_{12}\phi_{\theta,0}-2\Psi_{22}\phi_{X,0}\right)-\phi_{\theta,T}^{2}(\hat{q}_{T}+\tilde{q}_{22,T})\phi_{X,0}+\tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\phi_{\theta,0}\right]\\
d_\theta &= \tilde{m}\Big[-\phi_{\theta,T}+\phi_{\theta,T}^{2}\left(\phi_{\theta,0}\left(2\Psi_{11}(\hat{q}_{T}+\tilde{q}_{22,T})+1\right)-\Psi_{12}(\hat{q}_{T}+\tilde{q}_{22,T})\phi_{X,0}\right)\\
&\quad+\tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\left(2\Psi_{22}\phi_{X,0}-\Psi_{12}\phi_{\theta,0}\right)\Big]\\
d_x &= \tilde{m}\left[-\phi_{X,T}+\tilde{q}_{22,T}\phi_{x,T}^{2}\left(2\Psi_{22}\phi_{X,0}-\Psi_{12}\phi_{\theta,0}\right)+\tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\left(2\Psi_{11}\phi_{\theta,0}-\Psi_{12}\phi_{X,0}\right)+\phi_{\theta,T}^{2}\phi_{X,0}\right]\\
\beta_q &= -\tilde{m}\tilde{q}_{22,T}\phi_{x,T}^{2}\left[\phi_{\theta,0}\left(2\hat{q}_{T}\Psi_{12}\phi_{X,0}-\phi_{\theta,0}\left(2\hat{q}_{T}\Psi_{11}+1\right)\right)-2\hat{q}_{T}\Psi_{22}\phi_{X,0}^{2}\right]\\
&\quad+\tilde{m}\phi_{\theta,T}^{2}(\hat{q}_{T}+\tilde{q}_{22,T})\phi_{X,0}^{2}-2\tilde{m}\tilde{q}_{22,T}\phi_{x,T}\phi_{\theta,T}\phi_{\theta,0}\phi_{X,0}.
\end{aligned}$$

Therefore, matching coefficients of $f\left(T,\tilde{\theta}_T,\tilde{x}_T\right)=-Term^u$ gives the following boundary conditions:

$$\begin{aligned}
f(T) &= \frac{d_0^2}{2\beta_q}+\frac{1}{2}\tilde{m}\left[-f_{x,0}^{2}\left(2\hat{q}_{T}\tilde{q}_{22,T}\Psi_{22}\phi_{x,T}^{2}+\phi_{\theta,T}^{2}(\hat{q}_{T}+\tilde{q}_{22,T})\right)+f_{\theta,0}^{2}\tilde{q}_{22,T}\phi_{x,T}^{2}\left(2\hat{q}_{T}\Psi_{11}+1\right)\right]\\
&\quad+\tilde{m}f_{\theta,0}f_{x,0}\tilde{q}_{22,T}\phi_{x,T}\left(\hat{q}_{T}\Psi_{12}\phi_{x,T}+\phi_{\theta,T}\right)+f(0)-\frac{1}{2}\ln\tilde{m}-\ln\left|\phi_{\theta,T}\right|\\
f_{\theta\theta}(T) &= \frac{d_\theta^2}{\beta_q}+\tilde{m}\phi_{\theta,T}^{2}\left[2\Psi_{22}\left(2\Psi_{11}(\hat{q}_{T}+\tilde{q}_{22,T})+1\right)-\Psi_{12}^{2}(\hat{q}_{T}+\tilde{q}_{22,T})\right]\\
f_\theta(T) &= \frac{d_0 d_\theta}{\beta_q}+\tilde{m}\phi_{\theta,T}\Big[f_{\theta,0}\left(\phi_{\theta,T}\left(2\Psi_{11}(\hat{q}_{T}+\tilde{q}_{22,T})+1\right)-\tilde{q}_{22,T}\Psi_{12}\phi_{x,T}\right)\\
&\quad+f_{x,0}\left(2\tilde{q}_{22,T}\Psi_{22}\phi_{x,T}-\Psi_{12}\phi_{\theta,T}(\hat{q}_{T}+\tilde{q}_{22,T})\right)\Big]
\end{aligned}$$

$$
\begin{aligned}
f_{xx}(T) &= \frac{d_x^2}{\beta_q} + \tilde{m}\left[2\Psi_{11}\phi_{\theta,T}^2 - \tilde{q}_{22,T}\left(\Psi_{12}^2 - 4\Psi_{11}\Psi_{22}\right)\phi_{x,T}^2\right] \\
f_x(T) &= \frac{d_0 d_x}{\beta_q} - \tilde{m}\left[f_{\theta,0}\tilde{q}_{22,T}\left(\Psi_{12}\phi_{x,T}^2 - 2\Psi_{11}\phi_{x,T}\phi_{\theta,T}\right) - f_{x,0}\left(2\tilde{q}_{22,T}\Psi_{22}\phi_{x,T}^2 - \tilde{q}_{22,T}\Psi_{12}\phi_{x,T}\phi_{\theta,T} + \phi_{\theta,T}^2\right)\right] \\
f_{\theta x}(T) &= \frac{d_\theta d_x}{\beta_q} + \tilde{m}\phi_{\theta,T}\left[\Psi_{12}\phi_{\theta,T} - \tilde{q}_{22,T}\left(\Psi_{12}^2 - 4\Psi_{11}\Psi_{22}\right)\phi_{x,T}\right]
\end{aligned}
$$

Market Clearing.

Market clearing requires $(1-\omega)\alpha + \omega\beta = \theta_T$. Hence,

$$
(1-\omega)\frac{c_0 + c_\theta\theta_T + c_x\hat{x}_T + c_\Delta\Delta_T}{r\alpha_q} + \omega\frac{d_0 + d_\theta\theta_T + d_x\hat{x}_T + \left(d_\theta\frac{\phi_{x,T}}{\phi_{\theta,T}} - d_x\right)\Delta_T}{r\beta_q} = \theta_T,
$$

which gives the additional condition of: $\frac{(1-\omega)c_\Delta}{\alpha_q} + \frac{\omega\left(d_\theta\frac{\phi_{x,T}}{\phi_{\theta,T}} - d_x\right)}{\beta_q} = 0$.

Overall, in the above general setup, the following coefficients system needs to be identified:

$$
\begin{aligned}
&\phi(t), \phi_\theta(t), \phi_x(t), \phi_\Delta(t) \\
&g(t), g_\theta(t), g_{\theta\theta}(t), g_x(t), g_{xx}(t), g_\Delta(t), g_{\Delta\Delta}(t), g_{\theta x}(t), g_{\theta\Delta}(t), g_{x\Delta}(t) \\
&f(t), f_\theta(t), f_{\theta\theta}(t), f_x(t), f_{xx}(t), f_{\theta x}(t).
\end{aligned}
$$

In order to simplify this complicated system, let's first start with an initial choice for $\tilde{q}_T$. In general, the exercise should be i) start from an initial guess for $\phi_x(t)$ and $\phi_\theta(t)$; ii) solve for the dynamics of $(\tilde{x}_t, \tilde{q}_t)$; iii) solve for the entire equilibrium; iv) update $\phi_x(t)$ and $\phi_\theta(t)$ and iterate on the above procedure. For now, let's look at a much less ambitious question. In the following exercise, take $\hat{q}_t$ and $\tilde{q}_t$ as given. Everything should be represented as functions of the fundamental parameters, $\hat{q}_t$ and $\tilde{q}_t$. Assume $g_x^{n+1}(0) = g_{xx}^{n+1}(0) = g_{\theta x}^{n+1}(0) = 0$, and $f_x^{n+1}(0) = f_{xx}^{n+1}(0) = f_{\theta x}^{n+1}(0) = 0$. Start with an initial guess for

$$
\begin{aligned}
&\phi^{n+1}(0), \phi_\theta^{n+1}(0), \phi_x^{n+1}(0), \phi_\Delta^{n+1}(0), \\
&g^{n+1}(0), g_\theta^{n+1}(0), g_{\theta\theta}^{n+1}(0), \\
&f^{n+1}(0), f_\theta^{n+1}(0), f_{\theta\theta}^{n+1}(0).
\end{aligned}
$$

List the set of equations that can be used to solve for

$$
\begin{aligned}
&\phi^n(T), \phi_\theta^n(T), \phi_x^n(T), \phi_\Delta^n(T) \\
&g^n(T), g_\theta^n(T), g_{\theta\theta}^n(T), g_x^n(T), g_{xx}^n(T), g_\Delta^n(T), g_{\Delta\Delta}^n(T), g_{\theta x}^n(T), g_{\theta\Delta}^n(T), g_{x\Delta}^n(T) \\
&f^n(T), f_\theta^n(T), f_{\theta\theta}^n(T), f_x^n(T), f_{xx}^n(T), f_{\theta x}^n(T).
\end{aligned}
$$

The question is, does the solution satisfy $g_x^n(T) = g_{xx}^n(T) = g_{\theta x}^n(T) = 0$, and $f_x^n(T) = f_{xx}^n(T) =$

$f_{\theta x}^{n}(T)=0$? First do some simplifications for the boundary conditions. Denote

$$
\begin{aligned}
l_0 &= \phi(T)-\phi(0); & l_x &= \phi_x(T)-\phi_x(0);\\
l_\Delta &= \phi_\Delta(T)-\phi_\Delta(0); & l_\theta &= \phi_\theta(T)-\phi_\theta(0);\\
l_y &= \phi_X(T)-\phi_X(0) = l_\Delta + l_x.
\end{aligned}
$$

Given $g_x^{n+1}(0)=g_{xx}^{n+1}(0)=g_{\theta x}^{n+1}(0)=0$, and $f_x^{n+1}(0)=f_{xx}^{n+1}(0)=f_{\theta x}^{n+1}(0)=0$,

$$
\begin{aligned}
\hat{m} &= 1;\ c_0=-l_0;\ c_\theta=-l_\theta;\ c_x=-l_y;\ c_\Delta=\phi_{\Delta,T};\ \alpha_q=\phi_{X,0}^2\hat{q}_T.\\
g(T) &= g(0)+\frac{l_0^2}{2\alpha_q};\ g_{\theta\theta}(T)=g_{\theta\theta}(0)+\frac{l_\theta^2}{\alpha_q};\ g_\theta(T)=g_\theta(0)+\frac{l_0 l_\theta}{\alpha_q};\\
g_{\Delta\Delta}(T) &= \frac{\phi_{\Delta,T}^2}{\alpha_q};\ g_\Delta(T)=-\frac{l_0\phi_{\Delta,T}}{\alpha_q};\ g_{\theta\Delta}(T)=-\frac{l_\theta\phi_{\Delta,T}}{\alpha_q};\\
g_{xx}(T) &= \frac{l_y^2}{\alpha_q};\ g_x(T)=\frac{l_0 l_y}{\alpha_q};\ g_{\theta x}(T)=-\frac{l_\theta l_y}{\alpha_q};\ g_{x\Delta}(T)=-\frac{l_y\phi_{\Delta,T}}{\alpha_q};
\end{aligned}
$$

$$
\begin{aligned}
\tilde{m} &= \frac{1}{\phi_{\theta,T}^2+f_{\theta\theta,0}\phi_{x,T}^2\tilde{q}_{22,T}};\\
d_0 &= -l_0-f_{\theta,0}\tilde{m}\phi_{x,T}^2\left(\phi_{\theta,0}-\phi_{X,0}\right)\tilde{q}_{22,T};\\
d_\theta &= \tilde{m}\left(-l_\theta-\phi_{\theta,0}+\phi_{\theta,0}\phi_{\theta,T}^2+f_{\theta\theta,0}\phi_{\theta,T}\phi_{X,0}\phi_{x,T}\tilde{q}_{22,T}\right);\\
d_x &= \tilde{m}\left[-l_y+\phi_{X,0}\left(\phi_{\theta,T}^2+f_{\theta\theta,0}\phi_{x,T}^2\tilde{q}_{22,T}-1\right)\right];\\
\beta_q &= \tilde{m}\left[-2\phi_{\theta,0}\phi_{\theta,T}\phi_{X,0}\phi_{x,T}\tilde{q}_{22,T}+\phi_{x,T}^2\left(\phi_{\theta,0}^2+f_{\theta\theta,0}\phi_{X,0}^2\hat{q}_T\right)\tilde{q}_{22,T}+\phi_{\theta,T}^2\phi_{X,0}^2\left(\hat{q}_T+\tilde{q}_{22,T}\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
f(T) &= f(0)+\frac{d_0^2}{2\beta_q}+\frac{f_{\theta,0}^2\tilde{m}\phi_{x,T}^2\tilde{q}_{22,T}-\ln\tilde{m}-\ln\left(\phi_{\theta,T}^2\right)}{2};\\
f_{\theta\theta}(T) &= \frac{d_\theta^2}{\beta_q}+f_{\theta\theta,0}\tilde{m}\phi_{\theta,T}^2;\ f_\theta(T)=\frac{d_0 d_\theta}{\beta_q}+f_{\theta\theta,0}\tilde{m}\phi_{\theta,T}^2;\\
f_{xx}(T) &= \frac{d_x^2}{\beta_q};\ f_x(T)=\frac{d_0 d_x}{\beta_q};\ f_{\theta x}(T)=\frac{d_\theta d_x}{\beta_q}.
\end{aligned}
$$

Therefore, $\{l_0,l_\theta,l_x,l_\Delta\}$ can be solved as functions of $\hat{q}_T,\tilde{q}_{22,T}$, and $f^{n+1}(0), f_\theta^{n+1}(0), f_{\theta\theta}^{n+1}(0)$, $g^{n+1}(0), g_\theta^{n+1}(0), g_{\theta\theta}^{n+1}(0)$, $\phi^{n+1}(0), \phi_\theta^{n+1}(0), \phi_x^{n+1}(0), \phi_\Delta^{n+1}(0)$. However, there are multiple solutions. It is hard to prove the necessary conditions for $g_x^n(T)=g_{xx}^n(T)=g_{\theta x}^n(T)=0$, and $f_x^n(T)=f_{xx}^n(T)=f_{\theta x}^n(T)=0$.

However, if $g_x(t)=g_{xx}(t)=g_{\theta x}(t)=0$ and $f_x(t)=f_{xx}(t)=f_{\theta x}(t)=0$, the equilibrium conditions in the interior (equations (30) and (31)) will imply $e_x(t)=0$. Hence, equation (68) gives $\phi_X(t)=\frac{\phi_D}{b+r}$, which is a constant. This further gives $l_y=0$. The above equations would imply $g_x^n(T)=g_{xx}^n(T)=g_{\theta x}^n(T)=0$.

So at least a sufficient condition for $f_x^n(T)=f_{xx}^n(T)=f_{\theta x}^n(T)=0$ from the above equations could be determined for $d_x = 0$. Given $l_y = 0$ and assuming $\phi_X(0) \neq 0$, $\phi_x(T) \neq 0$ (or $\phi_\theta(T) \neq \pm 1$, $\tilde{q}_{22,T} \neq 0$), there exists a unique solution for $d_x = 0$:

$$f_{\theta\theta}(0) = \frac{1 - \phi_\theta^2(T)}{\phi_x^2(T)\,\tilde{q}_{22,T}}.$$

Given the conditions, the boundary system can be rewritten as

$$\begin{aligned}
\hat{m} &= 1;\ c_0 = -l_0;\ c_\theta = -l_\theta;\ c_\Delta = \phi_{\Delta,T};\ \alpha_q = \phi_X^2 \hat{q}_T. \\
\tilde{m} &= 1; \\
d_0 &= -l_0 + f_{\theta,0}\phi_{x,T}\left[\phi_X\phi_{\theta,T} + \left(l_\theta - \phi_{\theta,T}\right)\phi_{x,T}\right]\tilde{q}_{22,T}; \\
d_\theta &= -\frac{\phi_{\theta,T}}{\phi_{x,T}}\left[\phi_X\left(\phi_{\theta,T}^2 - 1\right) + \phi_{x,T} + \left(l_\theta - \phi_{\theta,T}\right)\phi_{\theta,T}\phi_{x,T}\right]; \\
\beta_q &= \phi_X^2\hat{q}_T + \left[\phi_X\phi_{\theta,T} + \left(l_\theta - \phi_{\theta,T}\right)\phi_{x,T}\right]^2\tilde{q}_{22,T}.
\end{aligned}$$

Using there equilibrium conditions,

$$\begin{aligned}
\phi_\theta(0) &= \frac{(\omega - 1)\left(\phi_X - \phi_{x,T}\right)}{\phi_X\omega\phi_{x,T}^2\hat{q}_T r} - \frac{\phi_X^2\hat{q}_T r}{\omega - 1} + \frac{\phi_X\left(\phi_{x,T} - \phi_X\right)\tilde{q}_{22,T} r}{\omega} \\
\phi_\theta(T) &= -\frac{\left(\phi_X - \phi_{x,T}\right)\left(1 - \omega + \phi_X^2\phi_{x,T}^2\hat{q}_T\tilde{q}_{22,T}r^2\right)}{\phi_X^2\omega\phi_{x,T}\hat{q}_T r} \\
l_0 &= -\frac{\phi_X f_{\theta,0}\omega\phi_{x,T}^2\hat{q}_T\tilde{q}_{22,T} r}{1 - \omega + \phi_X^2\phi_{x,T}^2\tilde{q}_{22,T} r^2}.
\end{aligned}$$

which corresponds to the simplified model in Section 4. □